Intersection numbers of Heegner divisors on Shimura curves

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In foundational papers, Gross, Zagier, and Kohnen established two formulas for arithmetic intersection numbers of certain Heegner divisors on integral models of modular curves. In [GZ1], only one imaginary quadratic discriminant plays a role. In [GZ2] and [GKZ], two quadratic discriminants play a role. In this paper we generalize the two-discriminant formula from the modular curves $X_0(N)$ to certain Shimura curves defined over $\mathbb{Q}$.

Our intersection formula was stated in [Ro], but the proof was only outlined there. Independently, the general formula was given, in a weaker and less explicit form, in [Ke2]; there it was proved completely. This paper is thus a synthesis of parts of [Ro] and [Ke2]. The intersection multiplicities computed here were used in [Ku] to derive a relation between height pairings and special values of the derivatives of certain Eisenstein series. We note also that Zhang [Zh] has generalized all of [GZ1] from ground field $\mathbb{Q}$ to general totally real ground fields F, working with general Shimura curves. So we certainly expect that all of [GKZ] should generalize similarly. Our work here can be viewed as a step in this direction.

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1 Eichler orders

Let Δ be a quaternion algebra over $\mathbb{Q}$ and let p be a prime. We begin by defining Eichler orders in $\Delta \otimes \mathbb{Q}_p$. Let $\mathcal{E}_p$ be an order in $\Delta \otimes \mathbb{Q}_p$ and let p^e be the reduced discriminant of $\mathcal{E}_p$. We say that $\mathcal{E}_p$ is an Eichler order of type $(p^e, 1)$ if $\mathcal{E}_p$ contains a subring isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. We say that $\mathcal{E}_p$ is an Eichler order of type $(1, p^e)$ if $\mathcal{E}_p$ contains a subring isomorphic to $\mathbb{Z}_{p^2}$, the ring of integers in the quadratic unramified extension of $\mathbb{Q}_p$. It is easily seen that two of these local Eichler orders are conjugate in $\Delta \otimes \mathbb{Q}_p$ if and only if they are of the same type. In fact, if $\mathcal{E}_p$ is an Eichler order of type $(p^e, 1)$ then $\Delta \otimes \mathbb{Q}_p \cong \mathbb{M}_2(\mathbb{Q}_p)$, and $\mathcal{E}_p$ is conjugate to the standard Eichler order

$$\hat{\mathcal{O}}_{p^e,1} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}_p) : p^e \mid c \right\}. \tag{1.1}$$

Let $\mathcal{M}_p$ be a maximal order in $\Delta \otimes \mathbb{Q}_p$, and let $\psi : \mathbb{Z}_{p^2} \to \mathcal{M}_p$ be an embedding. If $\mathcal{E}_p$ is an Eichler order of type $(1, p^e)$ then $\mathcal{E}_p$ is conjugate to the standard Eichler order

$$\hat{\mathcal{O}}_{1,p^e} = \psi(\mathbb{Z}_{p^2}) + p^f \mathcal{M}_p, \tag{1.2}$$

where $f = \lfloor e/2 \rfloor$. Note that if e is even then $\Delta \otimes \mathbb{Q}_p \cong \mathbb{M}_2(\mathbb{Q}_p)$, while if e is odd then $\Delta \otimes \mathbb{Q}_p$ is a division ring. For future use we let $\hat{B}_p$ denote the quaternion division ring whose maximal order is $\hat{\mathcal{O}}_{1,p}$.

We also need to define the notion of an orientation on a local Eichler order $\mathcal{E}_p$, which we assume is not of type $(1,1)$. For $e \geq 1$ define rings $R_{p^e,1} = (\mathbb{Z}/p^e\mathbb{Z}) \oplus (\mathbb{Z}/p^e\mathbb{Z})$ and $R_{1,p^e} = \mathbb{Z}_{p^2}/p^e\mathbb{Z}_{p^2}$. If $\mathcal{E}_p$ is an Eichler order of type $(p^e, 1)$ then an orientation on $\mathcal{E}_p$ is defined to be a ring homomorphism $\psi_p : \mathcal{E}_p \to R_{p^e,1}$. If $\mathcal{E}_p$ is an Eichler order of type $(1, p^e)$ then an orientation on $\mathcal{E}_p$ is defined to be a ring homomorphism $\psi_p : \mathcal{E}_p \to R_{1,p^e}$. Thus if $\mathcal{E}_p$ is an Eichler order in $\Delta \otimes \mathbb{Q}_p$ which is not isomorphic to $\mathbb{M}_2(\mathbb{Z}_p)$ there are exactly two orientations on $\mathcal{E}_p$. The usefulness of giving orientations to our Eichler orders may be summarized in the statement that the automorphisms of the oriented order $(\mathcal{E}_p, \psi_p)$ are precisely the maps given by conjugation by elements of $\mathcal{E}_p^\times$.

To define global Eichler orders we let $N^+ = \prod p^{n_p^+}$ and $N^- = \prod p^{n_p^-}$ be relatively prime positive integers and set $N = N^+N^-$. We say that an order $\mathcal{E}$ in Δ is an Eichler order of type (N^+, N^-) if $\mathcal{E} \otimes \mathbb{Z}_p$ is an Eichler order of type $(p^{n_p^+}, p^{n_p^-})$ for every prime p. An orientation on $\mathcal{E}$ consists of a collection $\{\psi_p\}_{p|N}$ of orientations on $\mathcal{E} \otimes \mathbb{Z}_p$ for every prime p which divides N.

Proposition 1.1 *Let Δ be a quaternion algebra over $\mathbb{Q}$ and let (N^+, N^-) be relatively prime positive integers.*

(a) Δ contains an Eichler order of type (N^+, N^-) if and only if $v_p(N^-)$ is odd precisely for those primes p which are ramified in Δ.

(b) Δ contains only finitely many isomorphism classes of oriented Eichler orders of type (N^+, N^-).

(c) If Δ is indefinite then Δ contains at most one isomorphism class of oriented Eichler orders of type (N^+, N^-).

Proof: (a) It follows from the definitions that if Δ contains an Eichler order of type (N^+, N^-) then $v_p(N^-)$ is odd if and only if Δ is ramified at p. On the other hand, if N^- satisfies this condition then one can easily construct an oriented Eichler order of type (N^+, N^-) from a maximal order in Δ.

(b) Let $(\mathcal{E}, \{\psi_p\}_{p|N})$ be an oriented Eichler order in Δ of type (N^+, N^-), let $\hat{\mathcal{E}} = \mathcal{E} \otimes \hat{\mathbb{Z}}$ be the profinite completion of $\mathcal{E}$, and let $\hat{\Delta} = \hat{\mathcal{E}} \otimes \mathbb{Q}$ be the ring of finite adèles of Δ. Associated to each $\beta = (\beta_p) \in \hat{\Delta}^\times$ there is a unique lattice L_β in Δ such that $L_\beta \otimes \mathbb{Z}_p = \beta_p(\mathcal{E} \otimes \mathbb{Z}_p)$ for all primes p. There is a bijection between the double coset space $S = \Delta^\times \backslash \hat{\Delta}^\times / \hat{\mathcal{E}}^\times$ and the set of all isomorphism classes of oriented Eichler orders of type (N^+, N^-), which associates to $\beta \in \hat{\Delta}^\times$ the pair $(\mathcal{E}_\beta, \{\phi_p^\beta\}_{p|N})$, where $\mathcal{E}_\beta$ is the left order of L_β and $\phi_p^\beta(x) = \psi_p(\beta_p^{-1} x \beta_p)$. Since S is finite [Vi, III, Cor. 5.5], the claim follows.

(c) By the strong approximation theorem [Vi, III, Th. 4.3] the reduced norm on $\hat{\Delta}^\times$ induces a bijection between S and the set $T = \mathbb{Q}^\times \backslash \hat{\mathbb{Q}}^\times / \mathrm{Nr}(\hat{\mathcal{E}}^\times)$. In fact $\mathrm{Nr}(\hat{\mathcal{E}}^\times) = \hat{\mathbb{Z}}^\times$, so T has just one element. $\square$

Corollary 1.2 *Let Δ be an indefinite quaternion algebra over $\mathbb{Q}$ ramified at the primes $p_1, \dots, p_s$. Set $N^- = p_1 p_2 \cdots p_s$, and let N^+ be a positive integer which is relatively prime to N^-. Then there are Eichler orders $\mathcal{O}_{1,N^-} \supset \mathcal{O}_{N^+,N^-}$ in Δ of types $(1, N^-)$ and (N^+, N^-) such that $\mathcal{O}_{1,N^-}/\mathcal{O}_{N^+,N^-}$ is a cyclic group of order N^+. The pair $(\mathcal{O}_{1,N^-}, \mathcal{O}_{N^+,N^-})$ is uniquely determined up to conjugacy in Δ.*

Proof: The existence of the pair $(\mathcal{O}_{1,N^-}, \mathcal{O}_{N^+,N^-})$ is clear; what must be proved is that all such pairs are conjugate in Δ. Since Δ is indefinite, it follows from Proposition 1.1(c) that $\mathcal{O}_{1,N^-}$ is determined uniquely up to conjugation. Let Σ be the set of Eichler orders $\mathcal{E}$ of type (N^+, N^-) such that $\mathcal{E} \subset \mathcal{O}_{1,N^-}$ and $\mathcal{O}_{1,N^-}/\mathcal{E}$ is cyclic of order N^+. For each prime p such that $p \mid N^+$ let Σ_p denote the set of local Eichler orders $\mathcal{E}_p$ of type $(p^{n^+}, 1)$ such that $\mathcal{E}_p \subset \mathbb{M}_2(\mathbb{Z}_p)$ and $\mathbb{M}_2(\mathbb{Z}_p)/\mathcal{E}_p$ is cyclic. Then $\mathbb{SL}_2(\mathbb{Z}_p)$ acts transitively by conjugation on Σ_p. Therefore by the strong approximation theorem the group of elements of $\mathcal{O}_{1,N^-}^\times$ with reduced norm 1 acts transitively by conjugation on Σ. It follows that the pair $(\mathcal{O}_{1,N^-}, \mathcal{O}_{N^+,N^-})$ is determined uniquely up to conjugation in Δ. $\square$

Let D_1, D_2 be negative integers which are squares (mod 4) such that $\mathbb{Q}(\sqrt{D_1}) \not\cong \mathbb{Q}(\sqrt{D_2})$. Let n be an integer such that $n \equiv D_1 D_2 \pmod 2$ and let B_n be the Clifford algebra of the binary quadratic form $q_n(x, y) = D_1 x^2 + 2nxy + D_2 y^2$. Thus B_n is a quaternion algebra over $\mathbb{Q}$ which is generated by elements e_1, e_2 such that $e_j^2 = D_j$ for $j = 1, 2$ and $e_1 e_2 + e_2 e_1 = 2n$. Let $g_j = (D_j + e_j)/2$ and let $S_n = \mathbb{Z}[g_1, g_2]$ be the subring of B_n generated by g_1 and g_2. Then

$$S_n = \mathbb{Z} + \mathbb{Z}g_1 + \mathbb{Z}g_2 + \mathbb{Z}g_1 g_2 \tag{1.3}$$

is an order in B_n with reduced discriminant $\delta_n = (n^2 - D_1 D_2)/4$. We may view S_n with the reduced norm form Nr as a quadratic space over $\mathbb{Z}$. By restricting Nr to

$L_n = \mathbb{Z} + \mathbb{Z}g_1 + \mathbb{Z}g_2$ we get a quadratic form

$$Q_n(x,y,z) = \mathrm{Nr}(x + yg_1 + zg_2) \tag{1.4}$$

$$= x^2 + \frac{D_1^2 - D_1}{4}y^2 + \frac{D_2^2 - D_2}{4}z^2 + D_1xy + D_2xz + \frac{D_1D_2 - n}{2}yz \tag{1.5}$$

with determinant $2\delta_n$.

Assume now that $n^2 < D_1D_2$ and $\gcd(D_1, D_2) = 1$. We factor the positive integer $-\delta_n$ into relatively prime factors δ_n^+, δ_n^- using the criterion

$$p \mid \delta_n^+ \text{ if } \left(\frac{D_j}{p}\right) = +1 \text{ for at least one } j = 1, 2, \tag{1.6}$$

$$p \mid \delta_n^- \text{ if } \left(\frac{D_j}{p}\right) = -1 \text{ for at least one } j = 1, 2, \tag{1.7}$$

where $\left(\frac{D_j}{p}\right)$ is the Kronecker symbol. Suppose $p \mid \delta_n$ and $p \nmid D_1D_2$. Then $D_1D_2 \equiv n^2 \pmod{4p}$, and hence $\left(\frac{D_1}{p}\right) = \left(\frac{D_2}{p}\right)$. Thus (1.6) and (1.7) uniquely determine the factorization $-\delta_n = \delta_n^+\delta_n^-$. For each prime p we have $p \nmid D_j$ for at least one $j \in \{1, 2\}$. Hence $\mathcal{O}_{D_j} \otimes \mathbb{Z}_p$ is isomorphic to either $\mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$. In particular, if $p \mid \delta_n^+$ then $\mathcal{O}_{D_j} \otimes \mathbb{Z}_p \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, and if $p \mid \delta_n^-$ then $\mathcal{O}_{D_j} \otimes \mathbb{Z}_p \cong \mathbb{Z}_{p^2}$. Since S_n contains a subring isomorphic to $\mathcal{O}_{D_j}$, it follows that S_n is an Eichler order of type (δ_n^+, δ_n^-).

2 Heegner divisors on Shimura curves

In this section we construct a scheme $\mathcal{X} = \mathcal{X}_{N^+,N^-,m}$ associated to the triple (N^+, N^-, m) for certain values of m. The scheme $\mathcal{X}$ is an integral model for a Shimura curve X which is defined over $\mathbb{Q}$. We also define Heegner divisors P_D on X and $\mathcal{P}_D$ on $\mathcal{X}$, where D is the discriminant of an order in an imaginary quadratic field.

The scheme $\mathcal{X}$ will be constructed as a moduli space for abelian surfaces A with additional structure. Part of the additional structure on A is a "special" $\mathcal{O}_{1,N^-}$-action, as defined by Drinfeld [Dr, §2A]. Let R be a ring, let A be an abelian surface over R, and let $i : \mathcal{O}_{1,N^-} \to \mathrm{End}_R(A)$ be an embedding. For $a \in \mathcal{O}_{1,N^-}$ let $\mathrm{Tr}(a) \in \mathbb{Z}$ denote the reduced trace of a, and let $\tau(a)$ denote the image of $\mathrm{Tr}(a)$ under the natural map $\mathbb{Z} \to R$. The embedding $i : \mathcal{O}_{1,N^-} \to \mathrm{End}_R(A)$ is said to be *special* if the trace of the action of $i(a)$ on $\mathrm{Lie}(A)$ is equal to $\tau(a)$ for all $a \in \mathcal{O}_{1,N^-}$. (If all the primes $p_1, \ldots, p_s$ which ramify in Δ are invertible in R then every embedding is special.) More generally, if Y is a scheme and A/Y is an abelian surface, we say that the embedding $i : \mathcal{O}_{1,N^-} \to \mathrm{End}_Y(A)$ is special if the induced map

$$i_R : \mathcal{O}_{1,N^-} \longrightarrow \mathrm{End}_R(A \times_Y \mathrm{Spec}(R)) \tag{2.1}$$

is special for every affine subscheme $\mathrm{Spec}(R)$ of Y.

We are interested in the moduli problem for isomorphism classes of triples (A, i, Z) over a scheme Y, where A/Y is an abelian surface, $i : \mathcal{O}_{1,N^-} \to \mathrm{End}_Y(A)$ is a special embedding, and Z is a subgroup scheme of A which is cyclic of order N^+ in the sense of [KM, 1.4]. Since this moduli problem is not representable, we will add a level-m structure to the problem for an appropriate value of m. The choice of m depends on $N = N^+N^-$ and on the imaginary quadratic discriminants D_1, D_2 which will be introduced in §3. Let m be a positive integer satisfying the following conditions:

$$\begin{aligned} &\text{C1: } \gcd(m, N) = 1. \\ &\text{C2: } m = m_1m_2 \text{ for some } m_1, m_2 \geq 4 \text{ such that } \gcd(m_1, m_2) = 1. \\ &\text{C3: } p > D_1D_2/4N \text{ for every prime } p \text{ which divides } m. \end{aligned} \tag{2.2}$$

Fix a ring isomorphism $\mathcal{O}_{1,N^-}/m\mathcal{O}_{1,N^-} \cong \mathbb{M}_2(\mathbb{Z}/m\mathbb{Z})$. Then $i : \mathcal{O}_{1,N^-} \to \mathrm{End}_Y(A)$ induces a ring homomorphism

$$i_m : \mathbb{M}_2(\mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathrm{End}_Y(A[m]), \tag{2.3}$$

where $A[m]$ denotes the m-torsion subgroup scheme of A. A $\Gamma_1(m)$-structure on the pair (A, i) is defined to be a point β in the kernel of $i_m\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$ which is defined over Y and has exact order m in the sense of [KM, 1.4].

Let $\mathcal{F}_m$ denote the functor from schemes to sets which associates to a scheme Y the set of isomorphism classes of 4-tuples (A, i, Z, β), where

1. A is an abelian surface defined over Y.
2. $i : \mathcal{O}_{1,N^-} \to \mathrm{End}_Y(A)$ is a special embedding.
3. Z is a cyclic subgroup scheme of A of order N^+ which is defined over Y and stabilized by $i(\mathcal{O}_{N^+,N^-})$.
4. β is a $\Gamma_1(m)$-structure on (A, i).

It follows from [Dr, Prop. 4.4] and [Bu, Lemma 2.2] that for $j = 1, 2$ the restriction of $\mathcal{F}_{m_j}$ to $\mathbb{Z}[1/m_j]$-schemes is represented by a scheme over $\mathbb{Z}[1/m_j]$. Therefore by [KM, 4.3.4] the restriction of $\mathcal{F}_m$ to $\mathbb{Z}[1/m_j]$-schemes is also represented by a $\mathbb{Z}[1/m_j]$-scheme. It follows that the functor $\mathcal{F}_m$ is represented by a scheme $\mathcal{X}$. By [Dr, Prop. 4.4], the scheme $\mathcal{X} \otimes \mathbb{Z}[1/m]$ is projective over $\mathbb{Z}[1/m]$. Hence $X := \mathcal{X} \otimes \mathbb{Q}$ is a projective curve over $\mathbb{Q}$.

Let k be a field, let $x \in \mathcal{X}(k)$, and let (A_x, i_x, Z_x, β_x) be the 4-tuple which corresponds to x. An endomorphism of the triple (A_x, i_x, Z_x) is defined to be an endomorphism of A_x which stabilizes Z_x and commutes with $i_x(a)$ for every $a \in \mathcal{O}_{1,N^-}$. Let D be a negative integer which is a square (mod 4), let $K = \mathbb{Q}(\sqrt{D})$, and let $\mathcal{O}_D = \mathbb{Z}[(D + \sqrt{D})/2]$ be the order of discriminant D in K. It follows from Proposition 2.1 below that there are only finitely many $x \in X(\mathbb{C})$ such that $\mathrm{End}(A_x, i_x, Z_x) \cong \mathcal{O}_D$. Therefore we may define

a divisor Q_D on X by setting $Q_D = \sum (x)$, where the sum is taken over all such x. It follows from the definition that Q_D is defined over $\mathbb{Q}$. Write $D = c^2 D_0$, where c is the conductor of $\mathcal{O}_D$ and D_0 is the discriminant of K. Define a divisor P_D on X by setting $P_D = \sum_{b|c} Q_{b^2 D_0}$. Then we have $P_D = \sum (x)$, where the sum is taken over points $x \in X(\mathbb{C})$ such that $\mathcal{O}_D$ embeds as a subring in $\mathrm{End}(A_x, i_x, Z_x)$. We call P_D the Heegner divisor of discriminant D on X. Since P_D is defined over $\mathbb{Q}$, we can also express P_D as a formal sum $P_D = \sum (y)$ of irreducible subschemes y of X. Let $\mathcal{P}_D$ be the divisor on $\mathcal{X}$ obtained by replacing each subscheme in this sum by its closure $\overline{y}$ in $\mathcal{X}$.

The following proposition gives a stringent condition that A must satisfy if (A, i, Z) corresponds to a point in the support of Q_D.

Proposition 2.1 *Let $x \in X(\mathbb{C})$ be a point in the support of Q_D. Let $\mathcal{R}$ be the smallest order in $K = \mathbb{Q}(\sqrt{D})$ which contains $\mathcal{O}_D$ and whose conductor is not divisible by any prime which is ramified in Δ. Then over $\mathbb{C}$ we have $A_x \cong E_1 \times E_2$, where E_1 and E_2 are elliptic curves such that $\mathrm{End}(E_1) \cong \mathrm{End}(E_2) \cong \mathcal{R}$.*

Proof: Since $\mathrm{End}(A_x, i_x, Z_x) \cong \mathcal{O}_D$ there is an embedding of $\Delta \otimes K$ into $\mathrm{End}(A_x) \otimes \mathbb{Q}$. Therefore by [Oo, Prop. 6.1] we see that K splits Δ and that

$$\mathrm{End}(A_x) \otimes \mathbb{Q} \cong \Delta \otimes K \cong \mathbb{M}_2(K). \tag{2.4}$$

It follows that $\mathrm{End}(A_x)$ is isomorphic to an order S in $\mathbb{M}_2(K)$, and that the complex points of A_x may be identified with a quotient $\mathbb{C}^2/L$, where L is a $\mathbb{Z}$-lattice in $K^2 \subset \mathbb{C}^2$. The stabilizer of L in $\mathbb{M}_2(K)$ is S, and hence for each prime p the stabilizer of $L \otimes \mathbb{Z}_p$ in $\mathbb{M}_2(K \otimes \mathbb{Q}_p)$ is $S \otimes \mathbb{Z}_p$.

The homomorphism $i_x : \mathcal{O}_{1,N^-} \to \mathrm{End}(A_x) \cong S$ induces a map

$$i_x \otimes \mathbb{Z}_p : \mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \longrightarrow S \otimes \mathbb{Z}_p. \tag{2.5}$$

If $p \nmid N^-$ then $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathbb{Z}_p)$, and hence $S \otimes \mathbb{Z}_p$ is isomorphic to an order in $\mathbb{M}_2(K \otimes \mathbb{Q}_p)$ which contains $\mathbb{M}_2(\mathbb{Z}_p)$. Such an order must be isomorphic to $\mathbb{M}_2(\mathcal{R}_p)$ for some order $\mathcal{R}_p$ in $K \otimes \mathbb{Q}_p$. Since $\mathrm{End}(A_x, i_x, Z_x) \cong \mathcal{O}_D$ we have $\mathcal{R}_p \cong \mathcal{O}_D \otimes \mathbb{Z}_p$.

If $p \mid N^-$ then p is ramified in Δ, and hence $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \hat{\mathcal{O}}_{1,p}$. Since K splits Δ we see that $K_p = K \otimes \mathbb{Q}_p$ is a field which is a quadratic extension of $\mathbb{Q}_p$. Let $\mathcal{O}_{K_p} = \mathcal{O}_K \otimes \mathbb{Z}_p$ be the ring of integers in K_p. We will show that $S \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathcal{O}_{K_p})$. Choose a $\mathbb{Q}_p$-embedding $\psi : K_p \to \Delta \otimes \mathbb{Q}_p$. We give $\Delta \otimes \mathbb{Q}_p$ the structure of a K_p-vector space by setting $a \cdot v = v\psi(a)$ for $a \in K_p$, $v \in \Delta \otimes \mathbb{Q}_p$. Left multiplication gives a representation of $\Delta \otimes \mathbb{Q}_p$ on this 2-dimensional K_p-vector space. On the other hand, since $S \otimes \mathbb{Z}_p$ is isomorphic to an order in $\mathbb{M}_2(K_p)$, the map $i_x \otimes \mathbb{Q}_p$ induces a representation of $\Delta \otimes \mathbb{Q}_p$ on K_p^2. By the Skolem-Noether theorem these two representations are isomorphic. Let $\Phi : K_p^2 \to \Delta \otimes \mathbb{Q}_p$ be a $(\Delta \otimes \mathbb{Q}_p)$-equivariant isomorphism of K_p-vector spaces. Since $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \hat{\mathcal{O}}_{1,p}$ stabilizes $L \otimes \mathbb{Z}_p$, it stabilizes $\Phi(L \otimes \mathbb{Z}_p)$ as well. Therefore $\Phi(L \otimes \mathbb{Z}_p)$ is a left $(\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p)$-ideal, and hence also a right $(\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p)$-ideal. Since $\psi(\mathcal{O}_{K_p}) \subset \mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$ this implies that $\Phi(L \otimes \mathbb{Z}_p)$ is an $\mathcal{O}_{K_p}$-module. Since $\Phi(L \otimes \mathbb{Z}_p)$ is free of rank 4 over $\mathbb{Z}_p$, it is free of rank 2 over $\mathcal{O}_{K_p}$. Therefore $L \otimes \mathbb{Z}_p$ is also a free

$\mathcal{O}_{K_p}$-module of rank 2. We conclude that the stabilizer $S \otimes \mathbb{Z}_p$ of $L \otimes \mathbb{Z}_p$ is isomorphic to $\mathbb{M}_2(\mathcal{O}_{K_p})$.

So far we have proved that $\operatorname{End}(A_x)$ is isomorphic to an order S in $\mathbb{M}_2(K)$ such that $S \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathcal{R}_p)$ for all p, where $\mathcal{R}_p = \mathcal{O}_{K_p}$ if p is ramified in Δ and $\mathcal{R}_p = \mathcal{O}_D \otimes \mathbb{Z}_p$ if p is not ramified in Δ. Hence the order $\mathcal{R}$ in the statement of the theorem is the unique order in K such that $\mathcal{R} \otimes \mathbb{Z}_p = \mathcal{R}_p$ for all p. To complete the proof we will show that S contains a nontrivial idempotent.

Let $L' \supset L$ be the $\mathcal{O}_K$-lattice generated by L. By choosing a new K-basis for $K^2 \subset \mathbb{C}^2$ we may assume that $L' = \mathcal{O}_K \oplus \mathcal{I}$ for some ideal $\mathcal{I} \subset \mathcal{O}_K$. The ideal $\mathcal{I}$ may be chosen to be relatively prime to every p such that $\mathcal{R}_p$ is not the maximal order in K_p. There is an abelian surface A' over $\mathbb{C}$ such that $A'(\mathbb{C}) \cong \mathbb{C}^2/L' \cong (\mathbb{C}/\mathcal{O}_K) \times (\mathbb{C}/\mathcal{I})$. The endomorphism ring of A' is

$$S' \cong \operatorname{End}_{\mathcal{O}_K}(L') \tag{2.6}$$

$$\cong \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, d \in \mathcal{O}_K,\, b \in \mathcal{I}^{-1},\, c \in \mathcal{I} \right\}, \tag{2.7}$$

which is a maximal order in $\mathbb{M}_2(K)$ containing S. There is an action of $\mathcal{O}_{1,N^-}$ on A' given by the map $i' : \mathcal{O}_{1,N^-} \to S'$ which is the composition of the inclusion $S \hookrightarrow S'$ with $i_x : \mathcal{O}_{1,N^-} \to S$.

The inclusion $L \hookrightarrow L'$ induces an $\mathcal{O}_{1,N^-}$-equivariant isogeny $\pi : A_x \to A'$. The kernel of π is a finite subgroup $G \cong L'/L$ of $A_x(\mathbb{C})$ which is stabilized by $i(\mathcal{O}_{1,N^-})$. Let $G \cong \oplus_p G_p$ be the decomposition of G into its p-primary components. Then $G_p \cong (L' \otimes \mathbb{Z}_p)/(L \otimes \mathbb{Z}_p)$, and $G_p = \{0\}$ for all p such that $\mathcal{R}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$. In particular, $G_p = \{0\}$ if p is ramified in Δ.

Let p be a prime such that $G_p \neq \{0\}$. Since $S \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathcal{R}_p)$ is the stabilizer of $L \otimes \mathbb{Z}_p$ in $\mathbb{M}_2(K \otimes \mathbb{Q}_p)$, we see that $L \otimes \mathbb{Z}_p$ is free of rank 2 over $\mathcal{R}_p$. Let $C_p \in \mathbb{M}_2(K \otimes \mathbb{Q}_p)$ be a matrix whose columns are $\mathcal{R}_p$-generators for $L \otimes \mathbb{Z}_p$. The columns of C_p also serve as $(\mathcal{O}_K \otimes \mathbb{Z}_p)$-generators for $L' \otimes \mathbb{Z}_p$, and by the assumption on $\mathcal{I}$ we have $L' \otimes \mathbb{Z}_p = (\mathcal{O}_K \otimes \mathbb{Z}_p)^2$. Therefore $C_p \in \mathbb{GL}_2(\mathcal{O}_K \otimes \mathbb{Z}_p)$. Let $J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. By multiplying one of the columns of C_p by $1/\det(C_p)$ we get a matrix $C'_p \in \mathbb{SL}_2(\mathcal{O}_K \otimes \mathbb{Z}_p)$ such that $C'_p J C'^{-1}_p = C_p J C_p^{-1}$ is a nontrivial idempotent which lies in $\operatorname{End}_{\mathcal{R}_p}(L \otimes \mathbb{Z}_p) = S \otimes \mathbb{Z}_p$.

For each p such that $G_p \neq \{0\}$ choose $n_p \geq 1$ such that p^{n_p} kills G_p. By the strong approximation theorem there exists a matrix $C \in \mathbb{SL}_2(\mathcal{O}_K)$ such that

$$C \equiv C'_p \pmod{p^{n_p}} \text{ for all } p \text{ such that } G_p \neq \{0\}, \tag{2.8}$$

$$C \equiv I_2 \pmod{\mathcal{I}}. \tag{2.9}$$

Then $e = CJC^{-1}$ is a nontrivial idempotent in S. Set $E_1 = eA_x$ and $E_2 = (1-e)A_x$. Then $A_x \cong E_1 \times E_2$ with $\operatorname{End}(E_1) \cong \operatorname{End}(E_2) \cong \mathcal{R}$. □

Let c be the conductor of $\mathcal{O}_D$.

Remark 2.2 Recall that the ray class field K_c of K with conductor $c\mathcal{O}_K$ is the maximum abelian extension of K whose ramification conductor divides $c\mathcal{O}_K$. The elliptic curves

E_1 and E_2 are defined over K_c, and hence the triple (A_x, i_x, Z_x) is defined over K_{cN^+}. Therefore x is rational over K_{cN^+}.

Remark 2.3 If $\gcd(c, N^-) \neq 1$ then by Proposition 2.1 we get $Q_D = 0$. Therefore if p divides $\gcd(c, N^-)$ then $P_D = P_{D/p^2}$. Hence we may assume without loss of generality that c is relatively prime to N^-.

Remark 2.4 Suppose $p^t \mid N^+$. Then the order in K which stabilizes $Z \subset E_1 \times E_2$ has conductor divisible by p^t. Hence if $p^t \nmid c$ and p is inert in K then $P_D = 0$.

Remark 2.5 Suppose $p \mid N^-$ and p splits in $K = \mathbb{Q}(\sqrt{D})$. Then $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \mathcal{O}_{1,p}$ cannot be embedded in $\mathbb{M}_2(K) \otimes \mathbb{Q}_p \cong \mathbb{M}_2(\mathbb{Q}_p \oplus \mathbb{Q}_p)$, so we have $P_D = 0$ in this case as well.

Associated to each point in the support of P_D is a collection of homomorphisms $\{\omega_p\}_{p|N}$ which is analogous to an orientation. The following well-known fact will be used to construct these homomorphisms.

Lemma 2.6 *Let R be a (possibly noncommutative) ring with 1 and let M be a free left R-module of rank 1 generated by $e \in M$. For $\phi \in \mathrm{End}_R(M)$ define $f(\phi) \in R$ by the formula $\phi(e) = f(\phi)e$. Then the map $f : \mathrm{End}_R(M) \to R^{op}$ is an isomorphism of rings, uniquely determined by M up to conjugation by units in R^{op}.*

Fix an orientation $\{\phi_p\}_{p|N}$ on $\mathcal{O}_{N^+,N^-}$ and let $x \in X(\mathbb{C})$ be a point in the support of P_D. Let p be a prime which divides N, let $T_p(A_x)$ be the p-adic Tate module of A_x, and let $U_p^x \supset T_p(A_x)$ be the lattice which corresponds to the p-primary part of Z_x. Since $T_p(A_x)$ is a left module over the maximal order $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$, it follows from [Re, Th. 18.7] that $T_p(A_x)$ is free of rank 1 over $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$. The $(\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p)$-module structure $i_x \otimes \mathbb{Z}_p$ on $T_p(A_x)$ induces an $(\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_p)$-module structure i_p on U_p^x. If $p \mid N^-$ it follows that $U_p^x = T_p(A_x)$ is free of rank 1 as a left module over $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_p = \mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$. If $p \mid N^+$ we may identify $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$ with $\mathbb{M}_2(\mathbb{Z}_p)$ and $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_p$ with the standard local Eichler order $\hat{\mathcal{O}}_{p^{n_p^+},1}$ defined in (1.1). Hence there is a generator e for $T_p(A_x)$ over $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathbb{Z}_p)$ such that $U_p^x/T_p(A_x)$ is generated by $\begin{bmatrix} p^{-n_p^+} & 0 \\ 0 & 0 \end{bmatrix} \cdot e$. It follows that U_p^x is free of rank 1 over $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_p$.

Let p be a prime which divides N. Since U_p^x is free of rank 1 over $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_p$, by Lemma 2.6 we get a ring isomorphism

$$\mathrm{End}(U_p^x, i_p) \longrightarrow \mathcal{O}_{N^+,N^-}^{op} \otimes \mathbb{Z}_p. \tag{2.10}$$

Since

$$\mathrm{End}(U_p^x, i_p) \cong \mathrm{End}(A_x, i_x, Z_x) \otimes \mathbb{Z}_p, \tag{2.11}$$

the orientation ϕ_p on $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_p$ induces a homomorphism

$$\omega_p^x : \mathrm{End}(A_x, i_x, Z_x) \otimes \mathbb{Z}_p \longrightarrow R_{p^{n_p^+},p^{n_p^-}}, \tag{2.12}$$

where $n_p^+ = v_p(N^+)$ and $n_p^- = v_p(N^-)$.

For each x in the support of P_D there are two embeddings of $\mathcal{O}_D$ into $\mathrm{End}(A_x, i_x, Z_x)$. Choose one of these and call it ρ_x. For $p \mid N$, $p \neq 2$ set $R_{(p)} = R_{p^{n_p^+}, p^{n_p^-}}$. By composing $\rho_x \otimes \mathbb{Z}_p$ with ω_p^x we get a homomorphism $\lambda_p^x : \mathcal{O}_D \otimes \mathbb{Z}_p \to R_{(p)}$. If $2 \mid N^+$ set $R_{(2)} = R_{2^{n_2^+ + 1}, 1}$, while if $2 \mid N^-$ set $R_{(2)} = R_{1, 2^{n^-+1}} = R_{1,4}$. In either case there is a ring homomorphism

$$\lambda_2^x : 1 \otimes \mathbb{Z}_2 + 2\mathcal{O}_D \otimes \mathbb{Z}_2 \longrightarrow R_{(2)} \tag{2.13}$$

defined by

$$\lambda_2^x(1 \otimes \alpha + 2\beta) = \alpha + 2 \cdot \omega_2^x \circ (\rho_x \otimes \mathbb{Z}_2)(\beta) \pmod{2^{v_2(N)+1}} \tag{2.14}$$

for $\alpha \in \mathbb{Z}_2$, $\beta \in \mathcal{O}_D \otimes \mathbb{Z}_2$. For every prime p such that $p \mid N$ set $a_p = v_p(2N)$. Then $\lambda_p^x(\sqrt{D})^2 \equiv D \pmod{p^{a_p}}$, and the ring homomorphism λ_p^x is determined by the value of $\lambda_p^x(\sqrt{D})$. Let ι_p denote the natural involution of the ring $R_{(p)}$. It follows from the definition of λ_p^x that $\iota_p(\lambda_p^x(\sqrt{D})) = -\lambda_p^x(\sqrt{D})$.

Assume that the conductor c of $\mathcal{O}_D$ is relatively prime to N. For each $p \mid N$ let b_p be an element of $R_{(p)}$ such that $b_p^2 \equiv D \pmod{p^{a_p}}$ and $\iota_p(b_p) = -b_p$. Since $p \nmid c$, there are two possibilities for b_p. Write $b = (b_p)_{p \mid N}$ and define

$$V_b = \{x \in \mathrm{Supp}(P_D) : \lambda_p^x(\sqrt{D}) = b_p \text{ for every } p \mid N\}. \tag{2.15}$$

Then we get a divisor $P_{D,b} = \sum_{x \in V_b} (x)$ on X such that $P_D = \sum_b P_{D,b}$. In general $P_{D,b}$ is not defined over $\mathbb{Q}$ and depends on the choices of the ρ_x, but the sum $P_{D,\pm b} = P_{D,b} + P_{D,-b}$ is a well-defined divisor over $\mathbb{Q}$. Define $\mathcal{P}_{D,\pm b}$ to be the closure of $P_{D,\pm b}$ in $\mathcal{X}$. Then $\mathcal{P}_{D,\pm b}$ is defined over $\mathbb{Z}$ and doesn't depend on the ρ_x. The divisors P_D and $\mathcal{P}_D$ have natural sum decompositions $P_D = \sum P_{D,\pm b}$ and $\mathcal{P}_D = \sum \mathcal{P}_{D,\pm b}$.

3 Intersection formulas

In this section we define the arithmetic intersection number $\langle \mathcal{Q}_1 \cdot \mathcal{Q}_2 \rangle_{\mathcal{X}}$ of two divisors $\mathcal{Q}_1, \mathcal{Q}_2$ on $\mathcal{X}$. We then give formulas for computing $\langle \mathcal{P}_{D_1, \pm b_1} \cdot \mathcal{P}_{D_2, \pm b_2} \rangle_{\mathcal{X}}$ and $\langle \mathcal{P}_{D_1} \cdot \mathcal{P}_{D_2} \rangle_{\mathcal{X}}$ in certain cases.

We wish to define a $\mathbb{Q}$-linear pairing $\langle \cdot \rangle_{\mathcal{X}}$ of divisors on $\mathcal{X}$ which intersect properly on regular points of $\mathcal{X}$. It suffices to define $\langle T_1 \cdot T_2 \rangle_{\mathcal{X}}$ for dimension-1 subschemes T_1, T_2 of $\mathcal{X}$ whose intersection is supported on a finite set of closed points of $\mathcal{X}$, each of which is regular. In this case we have $T_1 \cap T_2 \cong \mathrm{Spec}\, R$ for some finite ring R, where $T_1 \cap T_2$ is understood to mean $T_1 \times_{\mathcal{X}} T_2$. The arithmetic intersection number of T_1 with T_2 is defined to be

$$\langle T_1 \cdot T_2 \rangle_{\mathcal{X}} = \log \# R. \tag{3.1}$$

This formula extends by $\mathbb{Z}$-linearity to give pairings of divisors on $\mathcal{X}$.

The following proposition implies that the intersection of $\mathcal{P}_{D_1}$ and $\mathcal{P}_{D_2}$ is supported on a finite set of closed points of $\mathcal{X}$.

Proposition 3.1 *Let t be a point on $\mathcal{X}$ which lies in the support of the intersection of $\mathcal{P}_{D_1}$ with $\mathcal{P}_{D_2}$ and let (A_t, i_t, Z_t, β_t) be the corresponding 4-tuple. Then t is a closed point of characteristic $p > 0$, and over $\overline{\mathbb{F}}_p$ we have $A_t \cong E \times E$ for any supersingular elliptic curve E.*

Proof: Since $\mathbb{Q}(\sqrt{D_1}) \not\cong \mathbb{Q}(\sqrt{D_2})$ the images of $\mathcal{O}_{D_1}$ and $\mathcal{O}_{D_2}$ generate a subalgebra of $\mathrm{End}(A_t, i_t, Z_t)$ with $\mathbb{Z}$-rank ≥ 4. Therefore by [Oo, Prop. 6.1], t is a point of characteristic $p > 0$, and A_t is isogenous to the product of two supersingular elliptic curves. Hence by Proposition 2.1, A_t is actually isomorphic to the product of two supersingular elliptic curves over $\overline{\mathbb{F}}_p$. Since $t \in \overline{x}$ for some x in the support of P_{D_1}, it follows that t is closed in $\mathcal{X}$. A theorem of Deligne [Shi, Th. 3.5] says that the isomorphism class of the product of two supersingular elliptic curves over $\overline{\mathbb{F}}_p$ does not depend on the factors. Therefore $A_t \cong E \times E$ for any supersingular elliptic curve E over $\overline{\mathbb{F}}_p$. $\square$

Let D_1, D_2 be negative integers which are squares (mod 4) such that $\mathbb{Q}(\sqrt{D_1}) \not\cong \mathbb{Q}(\sqrt{D_2})$. For $i = 1, 2$ let c_i be the conductor of the order $\mathcal{O}_{D_i}$. For each prime l such that $l \mid N$ we make the following additional assumptions:

$$l \text{ divides neither } c_1 \text{ nor } c_2, \text{ and} \tag{3.2}$$

$$l \text{ divides at most one of } D_1, D_2. \tag{3.3}$$

Assumption (3.2) guarantees that P_{D_1} and P_{D_2} are Heegner divisors in the sense of [Bi], and in any case may be made without loss of generality for $l \mid N^-$ by Remark 2.3. Assumption (3.3) implies that every point in the support of the intersection of $\mathcal{P}_{D_1}$ with $\mathcal{P}_{D_2}$ is regular (see Corollary 6.3). It follows that $\langle \mathcal{P}_{D_1} \cdot \mathcal{P}_{D_2} \rangle_{\mathcal{X}}$ is well-defined.

Let p be a prime. If p is unramified in Δ let $\Delta(p)$ denote the quaternion algebra over $\mathbb{Q}$ which is ramified at $\infty, p, p_1, \ldots, p_s$. If $p = p_j$ is ramified in Δ let $\Delta(p)$ denote the quaternion algebra over $\mathbb{Q}$ which is ramified at $\infty, p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_s$. Let $\mathcal{S}_p$ denote the set of Eichler orders $\mathcal{E} \subset \Delta(p)$ of type (N^+, pN^-); it follows from Proposition 1.1(a) that $\mathcal{S}_p$ is not empty. We view the elements of $\mathcal{S}_p$ as lattices in $\Delta(p)$ with the $\mathbb{Z}$-valued quadratic forms induced by the reduced norm form on $\Delta(p)$. The assumption that $\mathcal{E}$ has type (N^+, pN^-) determines the isometry class of the reduced norm form on $\mathcal{E} \otimes \mathbb{Z}_l$ for every finite prime l, and $\mathcal{E} \otimes \mathbb{R} \cong \Delta(p) \otimes \mathbb{R}$ is positive definite since $\Delta(p)$ is ramified at ∞. Therefore every $\mathcal{E} \in \mathcal{S}_p$ belongs to the same genus of quadratic spaces. Let $\mathcal{G}_p$ be a set of representatives of the proper equivalence classes of this genus.

Let n be an integer such that $n^2 < D_1 D_2$ and $n \equiv D_1 D_2 \pmod 2$. Let L be a quadratic space over $\mathbb{Z}$ with finite rank and let w_L denote the number of proper self-isometries of L. Define $\mathcal{R}_L(Q_n)$ to be the number of representations on L of the quadratic form Q_n defined in (1.4). Also let r be the number of distinct prime divisors of N, and set $\eta(m) = \frac{1}{2} m^2 \cdot \prod_{p \mid m} (1 - p^{-2})$.

We now state the first version of our intersection formula.

Theorem 3.2 *Let D_1, D_2 satisfy assumptions (3.2) and (3.3), and assume that m satisfies conditions C1, C2, and C3 in (2.2). Then the arithmetic intersection number of*

$\mathcal{P}_{D_1}$ with $\mathcal{P}_{D_2}$ on $\mathcal{X}$ is given by

$$\langle \mathcal{P}_{D_1} \cdot \mathcal{P}_{D_2} \rangle_{\mathcal{X}} = 2^{r-1} \cdot \eta(m) \cdot \sum_{p<\infty} \left(\sum_{\substack{n^2 < D_1 D_2 \\ n^2 \equiv D_1 D_2 \ (4pN)}} \left(\sum_{L \in \mathcal{G}_p} \frac{\mathcal{R}_L(Q_n)}{w_L} \right) \cdot \alpha_p(Q_n) \right) \cdot \log p, \quad (3.4)$$

where the local intersection multiplicities $\alpha_p(Q_n)$ are computed in (6.2) and (6.6).

Remark 3.3 The inner sum on the right side of (3.4) is a representation number in the sense of Siegel (see for instance [Ca, p. 377]).

Remark 3.4 Let k be an algebraically closed field whose characteristic does not divide m, and let (A, i) be an abelian surface with special $\mathcal{O}_{1,N^-}$-embedding defined over k. Then the pair (A, i) admits $2\eta(m)$ different $\Gamma_1(m)$-structures.

By strengthening assumption (3.3) we get a formula for $\langle \mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2} \rangle_{\mathcal{X}}$ which is stated explicitly in terms of finite Dirichlet series.

Definition 3.5 *For p prime and $e \geq 0$ define*

$$L_{p^e,1}(s) = 1 + p^{-s} + p^{-2s} + \cdots + p^{-es} \quad (3.5)$$
$$L_{1,p^e}(s) = 1 - p^{-s} + p^{-2s} - \cdots + (-1)^e p^{-es}. \quad (3.6)$$

For relatively prime positive integers M^+, M^- define

$$L_{M^+,M^-}(s) = \prod_{p^e \| M^+} L_{p^e,1}(s) \cdot \prod_{p^e \| M^-} L_{1,p^e}(s). \quad (3.7)$$

For each prime p such that $p \mid N$ set $a_p = v_p(2N)$. Since $\iota_p((b_j)_p) = -(b_j)_p$ we have $(b_1)_p (b_2)_p \equiv h_p \pmod{p^{a_p}}$ for some $h_p \in \mathbb{Z}$. Let h be an integer such that

$$h \equiv h_p \quad \pmod{p^{a_p}} \quad \text{for all } p \mid N, \quad (3.8)$$
$$h \equiv D_1 D_2 \quad \pmod{2} \quad \text{if } 2 \nmid N. \quad (3.9)$$

These congruences determine the class of $h \pmod{2N}$.

Theorem 3.6 *Assume that D_1, D_2 satisfy assumption (3.2), $\gcd(D_1, D_2) = 1$, and $\mathcal{P}_{D_j,\pm b_j} \neq 0$ for $j = 1, 2$. Assume further that m satisfies conditions C1, C2, and C3 in (2.2). Then the arithmetic intersection number of $\mathcal{P}_{D_1,\pm b_1}$ with $\mathcal{P}_{D_2,\pm b_2}$ on $\mathcal{X}$ is given by*

$$\langle \mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2} \rangle_{\mathcal{X}} = \eta(m) \cdot \sum_{\substack{n^2 < D_1 D_2 \\ n \equiv \pm h \ (2N)}} L'_{\delta_n^+/N^+, \delta_n^-/N^-}(0), \quad (3.10)$$

where $\pm h = \pm h(\pm b_1, \pm b_2)$ is determined by (3.8) and (3.9), and δ_n^+, δ_n^- are determined by (1.6) and (1.7).

Remark 3.7 If $p \mid N^+$ then since $\mathcal{P}_{D_j,\pm b_j} \neq 0$, Remark 2.5 implies that p is not inert in $\mathbb{Q}(\sqrt{D_j})$. Similarly, Remark 2.5 implies that if $p \mid N^-$ then p is not split in $\mathbb{Q}(\sqrt{D_j})$. Since $N \mid \delta_n$ for every n such that $n \equiv \pm h \pmod{2N}$, we get $N^+ \mid \delta_n^+$ and $N^- \mid \delta_n^-$. Hence the right side of (3.10) is well-defined.

For each value of $\pm h \pmod{2N}$ such that $h^2 \equiv D_1D_2 \pmod{4N}$ there are 2^{r-1} pairs $(\pm b_1, \pm b_2)$ such that $\pm b_1 b_2 \equiv \pm h \pmod{2N}$. Therefore by summing (3.10) over all $(\pm b_1, \pm b_2)$ we get a formula for $\langle \mathcal{P}_{D_1} \cdot \mathcal{P}_{D_2} \rangle_{\mathcal{X}}$.

Corollary 3.8 *Assume that D_1, D_2 satisfy assumption (3.2),* $\gcd(D_1, D_2) = 1$, *and m satisfies conditions C1, C2, and C3 in (2.2). Then*

$$\langle \mathcal{P}_{D_1} \cdot \mathcal{P}_{D_2} \rangle_{\mathcal{X}} = 2^{r-1} \cdot \eta(m) \cdot \sum_{\substack{n^2 < D_1D_2 \\ n^2 \equiv D_1D_2 \ (2N)}} L'_{\delta_n^+/N^+, \delta_n^-/N^-}(0). \tag{3.11}$$

4 Intersection points

Let t be a point in the support of the intersection of $\mathcal{P}_{D_1}$ with $\mathcal{P}_{D_2}$, and let (A_t, i_t, Z_t, β_t) be the corresponding 4-tuple. In this section we study the endomorphism ring of the triple (A_t, i_t, Z_t). In particular, we show that $\mathrm{End}(A_t, i_t, Z_t)$ is an Eichler order, and we construct an orientation on $\mathrm{End}(A_t, i_t, Z_t)$ which is induced by the orientation $\{\phi_l\}_{l \mid N}$ on $\mathcal{O}_{N^+,N^-}$.

Let T_1, T_2 be dimension-1 subschemes of $\mathcal{X}$ whose intersection is supported on a finite set of regular closed points of $\mathcal{X}$. Recall that the arithmetic intersection number of T_1 with T_2 is defined to be $\langle T_1 \cdot T_2 \rangle_{\mathcal{X}} = \log \#R$, where $T_1 \cap T_2 \cong \operatorname{Spec} R$. In practice, we will compute $\langle T_1 \cdot T_2 \rangle_{\mathcal{X}}$ as a sum

$$\langle T_1 \cdot T_2 \rangle_{\mathcal{X}} = \sum_{p < \infty} (T_1 \cdot T_2)_p \cdot \log p, \tag{4.1}$$

where $(T_1 \cdot T_2)_p$ is the intersection multiplicity of T_1 with T_2 at points of characteristic p. Thus $(T_1 \cdot T_2)_p$ is equal to the length of the $\mathbb{Z}_p$-module $R \otimes \mathbb{Z}_p$. Let $W_p = W(\overline{\mathbb{F}}_p)$ denote the ring of Witt vectors with coefficients in $\overline{\mathbb{F}}_p$. Then $(T_1 \cdot T_2)_p$ may also be computed as the length of the W_p-module $R \otimes W_p$, or as the intersection multiplicity of $T_1 \otimes W_p$ with $T_2 \otimes W_p$ on $\mathcal{X} \otimes W_p$.

Let t be a point of characteristic p in the support of the intersection of $\mathcal{P}_{D_1} \otimes W_p$ with $\mathcal{P}_{D_2} \otimes W_p$ on $\mathcal{X} \otimes W_p$. Then t is rational over the residue field $\overline{\mathbb{F}}_p$ of W_p and thus may be viewed as an element of $\mathcal{X}(\overline{\mathbb{F}}_p)$. Let E be a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ and set $\Lambda = \mathrm{End}(E)$. Then by Proposition 3.1 we have $A_t \cong E \times E$, and hence $\mathrm{End}(A_t) \cong \mathbb{M}_2(\Lambda)$. We will assume that E is defined over $\mathbb{F}_p$; this implies that the elements of $\mathrm{End}(E)$ are defined over $\mathbb{F}_{p^2}$. It is well-known that $H_p = \Lambda \otimes \mathbb{Q}$ is a quaternion algebra over $\mathbb{Q}$ which is ramified at p and ∞, and that Λ is a maximal order in H_p. It follows from the definition of $\Delta(p)$ that there is an embedding $h : \Delta(p) \to \mathbb{M}_2(H_p)$ such that $h(\Delta(p))$ is the commutant of $i(\Delta)$ in $\mathbb{M}_2(H_p)$. The endomorphism ring of the triple

(A_t, i_t, Z_t) consists of those elements of $\mathbb{M}_2(\Lambda)$ which commute with every element of $i_t(\mathcal{O}_{1,N^-})$ and stabilize Z_t. Therefore $\mathrm{End}(A_t, i_t, Z_t)$ is identified via h^{-1} with an order $\mathcal{E}$ in $\Delta(p)$.

We now determine the completions of $\mathrm{End}(A_t, i_t, Z_t) \cong \mathcal{E}$ at the finite places of $\mathbb{Q}$. Let $l \neq p$, let $T_l(A_t)$ denote the l-adic Tate module of A_t, and let $U_l^t \supset T_l(A_t)$ be the lattice which corresponds to the l-primary part of Z_t. As in §2, U_l^t is a free $(\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l)$-module of rank 1. Using Lemma 2.6 we get an isomorphism

$$\mathrm{End}(A_t, i_t, Z_t) \otimes \mathbb{Z}_l \cong \mathcal{O}_{N^+,N^-}^{op} \otimes \mathbb{Z}_l. \tag{4.2}$$

Therefore if $l \neq p$ then $\mathrm{End}(A_t, i_t, Z_t) \otimes \mathbb{Z}_l$ is a local Eichler order of type $(l^{n_l^+}, l^{n_l^-})$.

We now consider the completion of $\mathrm{End}(A_t, i_t, Z_t)$ at p. Leaving the subgroup Z_t aside we see that $\mathrm{End}(A_t, i_t) \otimes \mathbb{Z}_p$ is the commutant of $i(\mathcal{O}_{1,N^-}) \otimes \mathbb{Z}_p$ in $\mathrm{End}(A_t) \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$. If $p \nmid N^-$ then $i_t(\mathcal{O}_{1,N^-}) \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathbb{Z}_p)$ stabilizes $\hat{\mathcal{O}}_{1,p} \oplus \hat{\mathcal{O}}_{1,p}$, and hence $i_t(\mathcal{O}_{1,N^-}) \otimes \mathbb{Z}_p = g\mathbb{M}_2(\mathbb{Z}_p)g^{-1}$ for some $g \in \mathbb{GL}_2(\hat{\mathcal{O}}_{1,p})$. Since $g\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})g^{-1} = \mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ and the commutant of $\mathbb{M}_2(\mathbb{Z}_p)$ in $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ is $\hat{\mathcal{O}}_{1,p} \cdot I_2$, it follows that

$$\mathrm{End}(A_t, i_t) \otimes \mathbb{Z}_p \cong g(\hat{\mathcal{O}}_{1,p} \cdot I_2)g^{-1} \cong \hat{\mathcal{O}}_{1,p}. \tag{4.3}$$

Thus if $p \nmid N$ then $\mathrm{End}(A_t, i_t, Z_t) \otimes \mathbb{Z}_p \cong \hat{\mathcal{O}}_{1,p}$. If $p \mid N^+$ we may assume that $p \nmid D_1$ by (3.3). Then Remark 2.4 implies that p is split in $\mathbb{Q}(\sqrt{D_1})$, which contradicts the fact that $\mathcal{O}_{D_1} \otimes \mathbb{Z}_p$ embeds in $\mathrm{End}(A_t, i_t, Z_t) \otimes \mathbb{Z}_p \cong \hat{\mathcal{O}}_{1,p}$. So in fact assumption (3.3) implies that there are no intersection points in characteristic p if $p \mid N^+$.

Finally, if $p \mid N^-$ we need to consider embeddings of $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \hat{\mathcal{O}}_{1,p}$ into $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ which are induced by special embeddings $i_t : \mathcal{O}_{1,N^-} \to \mathrm{End}(A_t)$. We can write $\hat{\mathcal{O}}_{1,p} = \mathbb{Z}_{p^2} + \mathbb{Z}_{p^2}\pi$, where $\mathbb{Z}_{p^2} \subset \hat{\mathcal{O}}_{1,p}$ is the ring of integers in an unramified quadratic extension of $\mathbb{Q}_p$, and π is an element of $\hat{\mathcal{O}}_{1,p}$ which normalizes $\mathbb{Z}_{p^2}$ and satisfies $\pi^2 = p$. Consider first the embedding of $\mathbb{Z}_{p^2}$ into $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$. It is not hard to show that since i_t is special this embedding is conjugate by an element of $\mathbb{GL}_2(\hat{\mathcal{O}}_{1,p})$ to the map

$$x \longmapsto \begin{bmatrix} x & 0 \\ 0 & x' \end{bmatrix}, \tag{4.4}$$

where $x' = \pi x \pi^{-1}$ is the Galois conjugate of x over $\mathbb{Q}_p$. The commutant in $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ of the image of (4.4) is

$$C = \left\{ \begin{bmatrix} a & b\pi \\ c\pi & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_{p^2} \right\}. \tag{4.5}$$

The image of π under $i_t \otimes \mathbb{Z}_p$ is a matrix of the form

$$\Pi = \begin{bmatrix} e\pi & f \\ g & h\pi \end{bmatrix}, \tag{4.6}$$

with $e, f, g, h \in \mathbb{Z}_{p^2}$ and $\Pi^2 = pI_2$. If $p \nmid f$ then $M = \begin{bmatrix} 1 & 0 \\ e\pi & f \end{bmatrix}$ is a unit in C, and $M\Pi M^{-1} = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$. Similarly, if $p \nmid g$ there is $M \in C^\times$ such that $M\Pi M^{-1} = \begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix}$.

If $p \mid f$ and $p \mid g$ then by an iterative procedure one constructs $M \in C^\times$ such that $M\Pi M^{-1} = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}$. Hence, up to conjugation by units in C, there are three possibilities for $(i_t \otimes \mathbb{Z}_p)(\pi)$, namely

$$\Pi_1 = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix} \qquad \Pi_3 = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}. \tag{4.7}$$

Corresponding to the matrices in (4.7) are embeddings i_p^1, i_p^2, and i_p^3 of $\hat{\mathcal{O}}_{1,p}$ into $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ such that $i_p^j(\pi) = \Pi_j$. If $i : \mathcal{O}_{1,N^-} \to \mathrm{End}(E \times E)$ is a special embedding then $i \otimes \mathbb{Z}_p$ is conjugate to exactly one of these embeddings. We say that i is of type j if $i \otimes \mathbb{Z}_p$ is conjugate to i_p^j. The commutant of the image of i_p^1 consists of matrices of the form (4.5) with $d = a$ and $c = pb$; the commutant of the image of i_p^2 consists of matrices of the form (4.5) with $d = a$ and $b = pc$; and the commutant of the image of i_p^3 consists of matrices of the form (4.5) with $a, b, c, d \in \mathbb{Z}_p$. It follows that for $j = 1, 2$ the commutant of the image of i_p^j in $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ is a local Eichler order of type $(1, p^2)$, while the commutant of the image of i_p^3 in $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ is a local Eichler order of type $(p, 1)$. By (3.3) and Remark 2.5 we may assume $\mathcal{O}_{D_1} \otimes \mathbb{Z}_p \cong \mathbb{Z}_{p^2}$. Since $\mathbb{Z}_{p^2}$ cannot be embedded into a local Eichler order of type $(p, 1)$, this implies that the third case does not occur. Therefore if $p \mid N^-$ then $\mathrm{End}(A_t, i_t, Z_t) \otimes \mathbb{Z}_p$ is a local Eichler order of type $(1, p^2)$.

The following definition characterizes the set of potential intersection points in characteristic p.

Definition 4.1 *Let p be a prime.*

1. *If $p \nmid N$ define $\mathcal{T}_p$ to be the set of isomorphism classes of triples (A, i, Z) over $\overline{\mathbb{F}}_p$ such that $A \cong E \times E$ for some supersingular elliptic curve E.*

2. *If $p \mid N^-$ define $\mathcal{T}_p$ to be the set of isomorphism classes of triples (A, i, Z) over $\overline{\mathbb{F}}_p$ such that $A \cong E \times E$ for some supersingular elliptic curve E and i is of type 1 or 2.*

3. *If $p \mid N^+$ define $\mathcal{T}_p$ to be the empty set.*

Remark 4.2 In Proposition 5.1 we will show that for $p \nmid N^+$ the set $\mathcal{T}_p$ parametrizes isomorphism classes of Eichler orders of type (N^+, pN^-). Thus for each prime p the set $\mathcal{T}_p$ is finite.

Let $p \nmid N^+$ and let $(A, i, Z) \in \mathcal{T}_p$. We now use the orientation $\{\phi_l\}_{l \mid N}$ on $\mathcal{O}_{N^+,N^-}$ to construct an orientation $\{\psi_l\}_{l \mid pN}$ on $\mathrm{End}(A, i, Z)$. For $l \neq p$ let $T_l(A)$ be the l-adic Tate module of A and let $U_l \supset T_l(A)$ be the lattice which corresponds to the l-primary part of Z. Then U_l is free of rank 1 as a left module over $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l$. Using Lemma 2.6 we get an isomorphism

$$\mathrm{End}(A, i, Z) \otimes \mathbb{Z}_l \cong \mathrm{End}(U_l, i \otimes \mathbb{Z}_l) \cong \mathcal{O}_{N^+,N^-}^{op} \otimes \mathbb{Z}_l. \tag{4.8}$$

Thus the orientation ϕ_l on $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l$ induces an orientation ψ_l on $\mathrm{End}(A, i, Z) \otimes \mathbb{Z}_l$.

It remains to construct an orientation on $\mathrm{End}(A, i, Z) \otimes \mathbb{Z}_p$. It follows from the computations above (cf. (4.3) and (4.5)) that $\mathrm{End}(A, i, Z)$ acts on $\mathrm{Lie}(A)$ through scalar multiplication by elements of $\mathbb{F}_{p^2}$. If $p \nmid N$ this action gives an orientation

$$\psi_p : \mathrm{End}(A, i, Z) \otimes \mathbb{Z}_p \longrightarrow \mathbb{F}_{p^2} \cong R_{1,p}. \tag{4.9}$$

If $p \mid N^-$ we note that $\mathbb{Z}_{p^2}/p\mathbb{Z}_{p^2} \cong \mathbb{F}_{p^2}$ embeds in $\mathcal{O}_{1,N^-} \otimes \mathbb{F}_p$. Let $e \in \mathrm{Lie}(A)$ be an eigenvector for $\mathbb{F}_{p^2}$ such that $(\mathcal{O}_{1,N^-} \otimes \mathbb{F}_p) \cdot v$ spans $\mathrm{Lie}(A)$. (It follows from (4.4) and (4.7) that v exists and is uniquely determined up to scalar multiplication.) For every $\alpha \in \mathrm{End}(A, i, Z)$ there is a unique $\beta \in \mathcal{O}_{1,N^-} \otimes \mathbb{F}_p$ such that $\alpha(v) = i(\beta) \cdot v$. Define $\overline{\psi}_p : \mathrm{End}(A, i, Z) \to R_{1,p}$ by setting $\overline{\psi}_p(\alpha) = \phi_p(\beta)$. Since $\mathrm{End}(A, i, Z) \otimes \mathbb{Z}_p$ is an Eichler order of type $(1, p^2)$, the homomorphism $\overline{\psi}_p$ lifts uniquely to an orientation

$$\psi_p : \mathrm{End}(A, i, Z) \otimes \mathbb{Z}_p \longrightarrow R_{1,p^2}. \tag{4.10}$$

Combining the above results we get the following proposition:

Proposition 4.3 *(a) Let t be a point of characteristic p in the support of the intersection of $\mathcal{P}_{D_1}$ with $\mathcal{P}_{D_2}$. Then $(A_t, i_t, Z_t) \in \mathcal{T}_p$.*
(b) Let $(A, i, Z) \in \mathcal{T}_p$ and let $\{\psi_l\}_{l|pN}$ be the orientation on $\mathrm{End}(A, i, Z)$ *induced by the orientation $\{\phi_l\}_{l|N}$ on $\mathcal{O}_{N^+,N^-}$. Then* $(\mathrm{End}(A, i, Z), \{\psi_l\}_{l|pN})$ *is an oriented Eichler order of type (N^+, pN^-).*

5 Families of Eichler orders

In this section we describe the relation between the endomorphism rings of elements of $\mathcal{T}_p$ and isomorphism classes of oriented Eichler orders of type (N^+, pN^-). It follows from Proposition 3.1 that if $(A, i, Z) \in \mathcal{T}_p$ then $A \cong E \times E$ for any supersingular elliptic curve E over $\overline{\mathbb{F}}_p$. As in §4 we assume that E is defined over $\mathbb{F}_p$. We also let $F \in \Lambda = \mathrm{End}(E)$ denote the Frobenius endomorphism of E.

Proposition 5.1 *(a) If $p \nmid N$ then the map $(A, i, Z) \mapsto$* $(\mathrm{End}(A, i, Z), \{\psi_l\}_{l|pN})$ *gives a bijection between $\mathcal{T}_p$ and the set of isomorphism classes of oriented Eichler orders of type (N^+, pN^-).*
(b) If $p \mid N^-$ then the map $(A, i, Z) \mapsto$ $(\mathrm{End}(A, i, Z), \{\psi_l\}_{l|pN})$ *gives a bijection between the set of $(A, i, Z) \in \mathcal{T}_p$ such that i has type 1, and the set of isomorphism classes of oriented Eichler orders of type (N^+, pN^-). The same statement holds for i of type 2.*

Proof: We first show that the given maps are onto. Let $(\mathcal{E}, \{\mu_l\}_{l|pN})$ be an oriented Eichler order of type (N^+, pN^-). Let $i_0 : \mathcal{O}_{1,N^-} \to \mathbb{M}_2(H_p)$ be an embedding. Since the commutant of $i_0(\mathcal{O}_{1,N^-})$ in $\mathbb{M}_2(H_p)$ is isomorphic to $\Delta(p)$, there is an embedding $h_0 : \mathcal{E} \to \mathbb{M}_2(H_p)$ such that $h_0(\mathcal{E})$ commutes with $i_0(\mathcal{O}_{1,N^-})$. We will show that there exists $g \in \mathbb{GL}_2(H_p)$ and $Z \subset E \times E$ such that $g i_0(x) g^{-1}$ is a special embedding of $\mathcal{O}_{1,N^-}$

into $\mathbb{M}_2(\Lambda) \cong \mathrm{End}(E \times E)$ and $gh_0(y)g^{-1}$ gives an isomorphism between the oriented orders $(\mathcal{E}, \{\mu_l\}_{l|pN})$ and $\mathrm{End}(E \times E, gi_0(x)g^{-1}, Z)$. Let l be a prime such that $l \mid N$ and $l \neq p$. Then $g^{-1}(T_l(E \times E))$ must satisfy the following conditions:

1. The lattice $g^{-1}(T_l(E \times E))$ is stabilized by $i_0(\mathcal{O}_{1,N^-})$ and $h_0(\mathcal{E})$.

2. The orientations ϕ_l and μ_l are compatible with respect to $g^{-1}(T_l(E \times E))$.

To identify an appropriate g we will first find lattices which satisfy these properties.

Suppose $l \mid N^+$. Then $\mathbb{M}_2(H_p) \otimes \mathbb{Q}_l \cong \mathbb{M}_4(\mathbb{Q}_l)$ acts on $\mathbb{Q}_l^4$, and hence i_0 and h_0 induce an action of $(\mathcal{E} \otimes \mathbb{Z}_l) \otimes (\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l)$ on $\mathbb{Q}_l^4$. Using the Skolem-Noether theorem we may identify $\mathcal{E} \otimes \mathbb{Z}_l$ with $\hat{\mathcal{O}}_{l^{n_l^+},1}$, $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l$ with $\mathbb{M}_2(\mathbb{Z}_l)$, and $\mathbb{Q}_l^4$ with $\mathbb{M}_2(\mathbb{Q}_l)$ in such a way that this action is isomorphic to $(A \otimes B) \cdot X = AXB^\iota$, where $A \in \hat{\mathcal{O}}_{l^{n_l^+},1}$, $B \in \mathbb{M}_2(\mathbb{Z}_l)$, $X \in \mathbb{M}_2(\mathbb{Q}_l)$, and ι is the canonical involution of $\mathbb{M}_2(\mathbb{Q}_l)$. It follows that every lattice in $\mathbb{Q}_l^4 \cong \mathbb{M}_2(\mathbb{Q}_l)$ which is stabilized by $(\mathcal{E} \otimes \mathbb{Z}_l) \otimes (\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l)$ is a $\mathbb{Q}_l^\times$-multiple of one of the lattices

$$L_j = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}_l) : l^j \mid c \text{ and } l^j \mid d \right\} \tag{5.1}$$

for $0 \leq j \leq n_l^+$.

We may assume further that the identification of $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l$ with $\mathbb{M}_2(\mathbb{Z}_l)$ maps $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l$ onto $\hat{\mathcal{O}}_{l^{n_l^+},1}$. Let $M_j \supset L_j$ be a lattice such that M_j/L_j is cyclic of order $l^{n_l^+}$ and M_j is stabilized by both $\mathcal{E} \otimes \mathbb{Z}_l$ and $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l$. (The lattice M_j corresponds to the l-primary part of Z.) Such a lattice M_j exists if and only if $j = 0$ or $j = n_l^+$. For those cases we have

$$M_0 = \left\{ \begin{bmatrix} a & l^{-n_l^+} b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_l \right\} \tag{5.2}$$

and $M_{n_l^+} = \hat{\mathcal{O}}_{l^{n_l^+},1}$. Note that ϕ_l induces opposite orientations on $\mathcal{E} \otimes \mathbb{Z}_l$ with respect to the lattices M_0 and $M_{n_l^+}$ (cf. (4.8)). Let $\mathcal{M}_l \subset \mathbb{M}_2(H_p) \otimes \mathbb{Q}_l$ denote the stabilizer of the lattice L_j $(j = 0, n_l^+)$ such that ϕ_l induces the orientation μ_l on $\mathcal{E} \otimes \mathbb{Z}_l$ with respect to M_j.

Suppose $l \mid N^-$ with $l \neq p$. Then $\mathbb{M}_2(H_p) \otimes \mathbb{Q}_l \cong \mathbb{M}_4(\mathbb{Q}_l)$ acts on $\mathbb{Q}_l^4$, and hence i_0 and h_0 induce an action of $(\mathcal{E} \otimes \mathbb{Z}_l) \otimes (\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l)$ on $\mathbb{Q}_l^4$. Since $\mathcal{E} \otimes \mathbb{Z}_l \cong \mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l \cong \hat{\mathcal{O}}_{1,l}$, by the Skolem-Noether theorem this action is isomorphic to $(a \otimes b) \cdot x = axb^\iota$, where $a, b \in \hat{\mathcal{O}}_{1,l}$, $x \in \hat{B}_l$, and ι is the canonical involution of $\hat{B}_l$. There are two $\mathbb{Q}_l^\times$-equivalence classes of lattices in $\hat{B}_l$ stabilized by $(\mathcal{E} \otimes \mathbb{Z}_l) \otimes (\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l)$. These are represented by $\hat{\mathcal{O}}_{1,l}$ and $\pi\hat{\mathcal{O}}_{1,l}$, where π is a uniformizer for $\hat{\mathcal{O}}_{1,l}$. The orientation ϕ_l on $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l$ induces opposite orientations on $\mathcal{E} \otimes \mathbb{Z}_l$ with respect to $\hat{\mathcal{O}}_{1,l}$ and $\pi\hat{\mathcal{O}}_{1,l}$. Choose $j = 0, 1$ so that ϕ_l induces the orientation μ_l on $\mathcal{E} \otimes \mathbb{Z}_l$ with respect to $\pi^j\hat{\mathcal{O}}_{1,l}$, and let $\mathcal{M}_l \subset \mathbb{M}_2(H_p) \otimes \mathbb{Q}_l$ denote the stabilizer of $\pi^j\hat{\mathcal{O}}_{1,l}$.

Suppose $p \nmid N$. Then by the Skolem-Noether theorem we may identify $\mathbb{M}_2(H_p) \otimes \mathbb{Q}_p$ with $\mathbb{M}_2(\hat{B}_p)$ in such a way that $i_0(\mathcal{O}_{1,N^-}) \otimes \mathbb{Z}_p = \mathbb{M}_2(\mathbb{Z}_p)$ and $h(\mathcal{E}) \otimes \mathbb{Z}_p = \hat{\mathcal{O}}_{1,p} \cdot I_2$.

Suppose $p \mid N^-$ and i has type 1. Then we may identify $\mathbb{M}_2(H_p) \otimes \mathbb{Q}_p$ with $\mathbb{M}_2(\hat{B}_p)$ so that

$$i_0(\mathcal{O}_{1,N^-}) \otimes \mathbb{Z}_p = \left\{ \begin{bmatrix} x & y \\ py' & x' \end{bmatrix} : x, y \in \mathbb{Z}_{p^2} \right\} \tag{5.3}$$

$$h_0(\mathcal{E}) \otimes \mathbb{Z}_p = \left\{ \begin{bmatrix} a & b\pi \\ pb\pi & a \end{bmatrix} : a, b \in \mathbb{Z}_{p^2} \right\} \tag{5.4}$$

and ϕ_p induces the orientation ψ_p on $\mathcal{E} \otimes \mathbb{Z}_p$. In either case we define $\mathcal{M}_p$ to be the subring of $\mathbb{M}_2(H_p) \otimes \mathbb{Q}_p$ which corresponds to $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ under this identification.

Let $\mathcal{M}$ be a maximal order in $\mathbb{M}_2(H_p)$ such that $\mathcal{M} \supset i_0(\mathcal{O}_{1,N^-})$, $\mathcal{M} \supset h_0(\mathcal{E})$, and $\mathcal{M} \otimes \mathbb{Z}_l = \mathcal{M}_l$ for all l such that $l \mid pN$. Let $\mathcal{C}$ denote the commutant of $i_0(\mathcal{O}_{1,N^-})$ in $\mathcal{M}$. It follows from the constructions above that if $l \mid pN$ then $\mathcal{C} \otimes \mathbb{Z}_l = h_0(\mathcal{E}) \otimes \mathbb{Z}_l$. In particular, if $l \mid N$ and $l \neq p$ then $\mathcal{C} \otimes \mathbb{Z}_l$ is a local Eichler order of type $(1, l^{n_l^-})$; if $p \nmid N$ then $\mathcal{C} \otimes \mathbb{Z}_p$ is a local Eichler order of type $(1, p)$; and if $p \mid N^-$ then $\mathcal{C} \otimes \mathbb{Z}_p$ is a local Eichler order of type $(1, p^2)$. Suppose $l \nmid pN$. Then $\mathcal{M} \otimes \mathbb{Z}_l \cong \mathbb{M}_4(\mathbb{Z}_l)$ and $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l \cong \mathbb{M}_2(\mathbb{Z}_l)$, and hence $\mathcal{C} \otimes \mathbb{Z}_l \cong \mathbb{M}_2(\mathbb{Z}_l)$. Combining these local results we conclude that $\mathcal{C}$ is an Eichler order of type $(1, pN^-)$.

It follows from Theorems 21.6 and 34.9 in [Re] that the maximal orders of $\mathbb{M}_2(H_p)$ are all conjugate. Therefore there exists $g \in \mathbb{GL}_2(H_p)$ such that $g\mathcal{M}g^{-1} = \mathbb{M}_2(\Lambda)$. Let $i(x) = gi_0(x)g^{-1}$ and $h(y) = gh_0(y)g^{-1}$ for $x \in \mathcal{O}_{1,N^-}$ and $y \in \mathcal{E}$. By the construction of $\mathcal{M}$ we see that $i(\mathcal{O}_{1,N^-})$ and $h(\mathcal{E})$ are contained in $\mathbb{M}_2(\Lambda) \cong \mathrm{End}(E \times E)$. In addition, if $p \mid N^-$ then it follows from (5.3) that $i : \mathcal{O}_{1,N^-} \to \mathrm{End}(E \times E)$ is a special embedding of type 1. Let Z be the unique cyclic subgroup of $E \times E$ of order N^+ which is stabilized by both $h(\mathcal{E})$ and $i(\mathcal{O}_{N^+,N^-})$. Let $\{\psi_l\}_{l|pN}$ be the orientation on $\mathcal{E} \cong \mathrm{End}(E \times E, i, Z)$ induced by $\{\phi_l\}_{l|N}$. Then by the constructions above we have $\psi_l = \mu_l$ for all l such that $l \mid N$. Thus if $p \mid N$ then h induces an isomorphism

$$(\mathcal{E}, \{\mu_l\}_{l|pN}) \cong (\mathrm{End}(E \times E, i, Z), \{\psi_l\}_{l|pN}) \tag{5.5}$$

as required. Suppose $p \nmid N$. If $\psi_p = \mu_p$ then (5.5) still holds, while if $\psi_p \neq \mu_p$ then (5.5) holds after we replace $i(x)$ with $Fi(x)F^{-1}$ and $h(x)$ with $Fh(x)F^{-1}$.

To prove that our map is one-to-one we need to show that if the endomorphism rings of the triples (A_1, i_1, Z_1) and (A_2, i_2, Z_2) are isomorphic as oriented orders then $(A_1, i_1, Z_1) \cong (A_2, i_2, Z_2)$. We may assume that $A_1 = A_2 = E \times E$. Then there is an oriented Eichler order $(\mathcal{E}, \{\psi_l\}_{l|pN})$ in $\Delta(p)$ of type (N^+, pN^-), and embeddings $h_j : \mathcal{E} \to \mathbb{M}_2(\Lambda)$ for $j = 1, 2$ which induce isomorphisms between the oriented orders $(\mathcal{E}, \{\psi_l\}_{l|pN})$ and $\mathrm{End}(E \times E, i_j, Z_j)$. We need to show there is $g \in \mathbb{GL}_2(\Lambda)$ such that $Z_2 = gZ_1$ and $i_2(x) = gi_1(x)g^{-1}$ for all $x \in \mathcal{O}_{1,N^-}$.

By the Skolem-Noether theorem there is $g \in \mathbb{GL}_2(H_p)$ such that $i_2(x) = gi_1(x)g^{-1}$ and $h_2(y) = gh_1(y)g^{-1}$ for all $x \in \mathcal{O}_{1,N^-}$ and $y \in \mathcal{E}$. We may assume that $g \in \mathbb{M}_2(\Lambda)$, and that $|\mathrm{Nr}(g)|$ is as small as possible. We claim that $g \in \mathbb{GL}_2(\Lambda)$. Let R be the order in $\mathbb{M}_2(\Lambda)$ generated by $i_2(\mathcal{O}_{1,N^-})$ and $h_2(\mathcal{E})$. The lattices Λ^2 and $g(\Lambda^2)$ are stabilized by $i_2(\mathcal{O}_{1,N^-}) = gi_1(\mathcal{O}_{1,N^-})g^{-1}$ and $h_2(\mathcal{E}) = gh_1(\mathcal{E})g^{-1}$, and hence also by R. Let $l \neq p$.

Then $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l$, $\mathcal{E} \otimes \mathbb{Z}_l$, and $R \otimes \mathbb{Z}_l$ stabilize $T_l = T_l(E \times E)$ and $g(T_l)$. Suppose $l \mid N$ and $l \neq p$. Then there is a lattice $U_l \supset T_l$ such that U_l/T_l is cyclic of order $l^{n_l^+}$ and both U_l and $g(U_l)$ are stabilized by $i_2(\mathcal{O}_{N^+,N^-}) \otimes \mathbb{Z}_l$ and $h_2(\mathcal{E}) \otimes \mathbb{Z}_l$. The orientations on $\mathcal{E} \otimes \mathbb{Z}_l \cong h_2(\mathcal{E}) \otimes \mathbb{Z}_l$ induced by the orientation ϕ_l on $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l \cong i_2(\mathcal{O}_{N^+,N^-}) \otimes \mathbb{Z}_l$ with respect to the lattices U_l and $g(U_l)$ are identical.

Suppose $l \mid N$ with $l \neq p$. We saw above that there are two $\mathbb{Q}_l^\times$-equivalence classes of pairs of lattices $M_l \supset L_l$ in $\mathbb{Q}_l^4$ such that M_l/L_l is cyclic of order $l^{n_l^+}$, M_l is stabilized by $\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l$ and $\mathcal{E} \otimes \mathbb{Z}_l$, and L_l is stabilized by $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l$ and $\mathcal{E} \otimes \mathbb{Z}_l$. The orientation ϕ_l induces opposite orientations on $\mathcal{E} \otimes \mathbb{Z}_l$ with respect to the two possibilities for M_l. Therefore $g(T_l)$ lies in the same class as T_l, so $g(T_l) = l^k T_l$ for some $k \geq 0$. Suppose $l \nmid pN$. Then $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_l \cong \mathcal{E} \otimes \mathbb{Z}_l \cong \mathbb{M}_2(\mathbb{Z}_l)$ and hence $R \otimes \mathbb{Z}_l \cong \mathbb{M}_4(\mathbb{Z}_l)$ stabilizes T_l and $g(T_l)$. It follows that $g(T_l) = l^k T_l$ for some $k \geq 0$. In both cases we have $g = l^k \cdot g_0$ for some $g_0 \in \mathbb{M}_2(\Lambda)$. By the minimality of $|\mathrm{Nr}(g)|$ we get $k = 0$, and hence $g(T_l) = T_l$. It follows that $g \in \mathbb{GL}_2(\Lambda \otimes \mathbb{Z}_l)$ for every $l \neq p$.

Suppose $p \nmid N$. Then $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathbb{Z}_p)$ and $\mathcal{E} \otimes \mathbb{Z}_p \cong \hat{\mathcal{O}}_{1,p}$, and hence $R \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$. Therefore $g(\Lambda^2) \otimes \mathbb{Z}_p$ is stabilized by $\mathbb{M}_2(\Lambda \otimes \mathbb{Z}_p)$, so $g(\Lambda^2) \otimes \mathbb{Z}_p = F^k \cdot (\Lambda^2 \otimes \mathbb{Z}_p)$ for some $k \geq 0$. It follows from the minimality of $|\mathrm{Nr}(g)|$ that $k = 0, 1$. However, if $g(\Lambda^2) \otimes \mathbb{Z}_p = F \cdot (\Lambda^2 \otimes \mathbb{Z}_p)$ then $g \in F \cdot \mathbb{GL}_2(\Lambda)$, and hence $i_1(x)$ and $i_2(x) = g i_1(x) g^{-1}$ can't both be of type 1, contrary to assumption. Hence $g \in \mathbb{GL}_2(\Lambda \otimes \mathbb{Z}_p)$.

Suppose $p \mid N^-$. It follows using (5.3) and (5.4) that we may identify $\mathbb{M}_2(\Lambda \otimes \mathbb{Z}_p)$ with $\mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ in such a way that $R \otimes \mathbb{Z}_p$ contains the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$. Since $g(\Lambda^2) \otimes \mathbb{Z}_p$ is an $\hat{\mathcal{O}}_{1,p}$-lattice in $(H_p \otimes \mathbb{Q}_p)^2 \cong \hat{B}_p^2$ which is stabilized by $R \otimes \mathbb{Z}_p$, it is $\mathbb{Q}_p^\times$-equivalent to one of the lattices $\hat{\mathcal{O}}_{1,p} \oplus \hat{\mathcal{O}}_{1,p}$, $\hat{\mathcal{O}}_{1,p} \oplus \pi\hat{\mathcal{O}}_{1,p}$, $\hat{\mathcal{O}}_{1,p} \oplus p\hat{\mathcal{O}}_{1,p}$, $\pi\hat{\mathcal{O}}_{1,p} \oplus \pi\hat{\mathcal{O}}_{1,p}$, $\pi\hat{\mathcal{O}}_{1,p} \oplus p\hat{\mathcal{O}}_{1,p}$, or $\pi\hat{\mathcal{O}}_{1,p} \oplus p\pi\hat{\mathcal{O}}_{1,p}$. However, since i_1 and i_2 are both special embeddings, $g(\Lambda^2) \otimes \mathbb{Z}_p$ is not $\mathbb{Q}_p^\times$-equivalent to either $\hat{\mathcal{O}}_{1,p} \oplus \pi\hat{\mathcal{O}}_{1,p}$ or $\pi\hat{\mathcal{O}}_{1,p} \oplus p\hat{\mathcal{O}}_{1,p}$. Since i_1 and i_2 are both of of type 1, $g(\Lambda^2) \otimes \mathbb{Z}_p$ is not equivalent to either $\hat{\mathcal{O}}_{1,p} \oplus p\hat{\mathcal{O}}_{1,p}$ or $\pi\hat{\mathcal{O}}_{1,p} \oplus \pi\hat{\mathcal{O}}_{1,p}$. Since ϕ_p induces the same orientation on $\mathcal{E} \otimes \mathbb{Z}_p$ with respect to the embedding pairs (i_1, h_1) and (i_2, h_2), $g(\Lambda^2) \otimes \mathbb{Z}_p$ is not equivalent to $\pi\hat{\mathcal{O}}_{1,p} \oplus p\pi\hat{\mathcal{O}}_{1,p}$. We conclude that $g(\Lambda^2)$ is $\mathbb{Q}_p^\times$-equivalent to $\hat{\mathcal{O}}_{1,p} \oplus \hat{\mathcal{O}}_{1,p}$. Hence $g(\Lambda^2) \otimes \mathbb{Z}_p = p^k \cdot (\Lambda^2 \otimes \mathbb{Z}_p)$ for some $k \geq 0$. By the minimality of $|\mathrm{Nr}(g)|$ we get $k = 0$, and hence $g \in \mathbb{GL}_2(\Lambda \otimes \mathbb{Z}_p)$.

Combining the above results we get $g \in \mathbb{GL}_2(\Lambda)$. Finally, the l-primary subgroup of Z_1 is the unique cyclic subgroup of $E \times E$ of order $l^{n_l^+}$ which is stabilized by both $h_1(\mathcal{E})$ and $i_1(\mathcal{O}_{N^+,N^-})$. A similar statement holds for $j = 2$, and so g maps the l-primary subgroup of Z_1 to the l-primary subgroup of Z_2. Therefore $Z_2 = gZ_1$. □

Recall that $\mathcal{G}_p$ is a set of representatives of the proper equivalence classes of a genus of quaternary quadratic spaces over $\mathbb{Z}$, and that $\mathcal{S}_p$ denotes the set of Eichler orders in $\Delta(p)$ of type (N^+, pN^-). We may view the elements of $\mathcal{S}_p$ as quadratic spaces over $\mathbb{Z}$ with the $\mathbb{Z}$-valued quadratic forms induced by the reduced norm on $\Delta(p)$.

Proposition 5.2 *(a) Every $L \in \mathcal{G}_p$ which represents 1 over $\mathbb{Z}$ is properly equivalent to some $\mathcal{E} \in \mathcal{S}_p$.*

(b) The orders $\mathcal{E}, \mathcal{E}' \in \mathcal{S}_p$ are properly equivalent quadratic spaces if and only if $\mathcal{E}' = a\mathcal{E}a^{-1}$ for some $a \in \Delta(p)^\times$.

To facilitate the proof we recall the following well-known fact (cf. [Vi, I, Th. 3.3]):

Lemma 5.3 *Let B be a quaternion algebra over a field F whose characteristic is not 2. Then every proper self-isometry of B has the form $\phi(x) = axb^{-1}$ for some $a, b \in B^\times$ such that $\mathrm{Nr}(a) = \mathrm{Nr}(b)$.*

Proof of Proposition 5.2: (a) Choose an Eichler order $\mathcal{E}_0 \subset \Delta(p)$ of type (N^+, pN^-). Then $\mathcal{E}_0$ with the reduced norm form belongs to the same genus as L, so by the weak Hasse principle [Ca, p. 76] L is isometric to a lattice J in $\Delta(p)$. Let l be a prime. Then $\mathcal{E}_0 \otimes \mathbb{Z}_l$ is isomorphic to one of the standard local Eichler orders given in (1.1) and (1.2). Since the standard local Eichler orders are stabilized by the appropriate canonical involutions, there is a *proper* isometry between $J \otimes \mathbb{Z}_l$ to $\mathcal{E}_0 \otimes \mathbb{Z}_l$. It follows from Lemma 5.3 that there are $a_l, b_l \in (\Delta(p) \otimes \mathbb{Q}_l)^\times$ such that $J \otimes \mathbb{Z}_l = a_l(\mathcal{E}_0 \otimes \mathbb{Z}_l)b_l^{-1}$.

Let $\mathcal{E}_J \subset \Delta(p)$ be the right order of J. Since $\mathcal{E}_J \otimes \mathbb{Z}_l = b_l(\mathcal{E}_0 \otimes \mathbb{Z}_l)b_l^{-1}$ for every prime l we have $\mathcal{E}_J \in \mathcal{S}_p$. Let $u \in J$ be such that $\mathrm{Nr}(u) = 1$. Then $u\mathcal{E}_J \subset J$, and for every prime l the quadratic spaces $u\mathcal{E}_J \otimes \mathbb{Z}_l$ and $J \otimes \mathbb{Z}_l$ are both isometric to the local Eichler order $\mathcal{E}_0 \otimes \mathbb{Z}_l$. Since $\mathcal{E}_0 \otimes \mathbb{Z}_p$ is a nondegenerate quadratic space, this implies $u\mathcal{E}_J \otimes \mathbb{Z}_l = J \otimes \mathbb{Z}_l$. Thus $u\mathcal{E}_J = J$, and hence J and L are isometric to $\mathcal{E}_J \in \mathcal{S}_p$. If the isometry $f : L \to \mathcal{E}_J$ is not proper, replace f with $\iota \circ f$ and $\mathcal{E}_J$ with $\iota(\mathcal{E}_J)$, where ι is the canonical involution of $\Delta(p)$.

(b) If $\mathcal{E}' = a\mathcal{E}a^{-1}$ with $a \in \Delta(p)^\times$ then $x \mapsto axa^{-1}$ is a proper isometry from $\mathcal{E}$ to $\mathcal{E}'$. Conversely, suppose $\phi : \mathcal{E} \to \mathcal{E}'$ is a proper isometry. By Lemma 5.3 we have $\phi(x) = axb^{-1}$ for some $a, b \in \Delta(p)^\times$. Clearly $\phi(1) = ab^{-1}$ is a unit in $\mathcal{E}'$, so $a\mathcal{E}a^{-1} = \phi(\mathcal{E})(ab^{-1})^{-1} = \mathcal{E}'$. □

6 Universal deformations

Let (A_t, i_t, Z_t, β_t) correspond to a point $t \in \mathcal{X}(\overline{\mathbb{F}}_p)$. Let $W_p = W(\overline{\mathbb{F}}_p)$ denote the ring of Witt vectors with coefficients in $\overline{\mathbb{F}}_p$, and let $\hat{\mathcal{X}}_t$ be the completion of $\mathcal{X} \otimes W_p$ at t. Since $\mathcal{X}$ is a fine moduli space, $\hat{\mathcal{X}}_t$ is a universal deformation space for (A_t, i_t, Z_t, β_t). In this section we study $\hat{\mathcal{X}}_t$. Let $\hat{A}_t$ be the formal group of A_t and let $\hat{\imath}_t : \mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \to \mathrm{End}(\hat{A}_t)$ be the map induced by i_t. To begin we show that $\hat{\mathcal{X}}_t$ is a universal deformation space for the pair $(\hat{A}_t, \hat{\imath}_t)$.

Lemma 6.1 *Let p be a prime such that $p \nmid mN^+$, and let R be a complete Noetherian local ring with residue field $\overline{\mathbb{F}}_p$. Then there are natural bijections between*

(a) The set of deformations over R of the 4-tuple (A_t, i_t, Z_t, β_t),

(b) The set of deformations over R of the pair (A_t, i_t), and

(c) The set of deformations over R of the pair $(\hat{A}_t, \hat{\imath}_t)$.

Therefore $\hat{\mathcal{X}}_t$ serves as a universal deformation space for either (A_t, i_t, Z_t, β_t), (A_t, i_t), or $(\hat{A}_t, \hat{\imath}_t)$.

Proof: Since $p \nmid mN^+$, Hensel's Lemma [Mi, I, Th. 4.2(d)] implies that if (A, i) is a deformation of (A_t, i_t) defined over R then Z_t extends uniquely to a subgroup scheme $Z \subset A$ which is finite and flat of order N^+ over R, and β_t extends uniquely to a $\Gamma_1(m)$-structure β on A. This gives a bijection between (a) and (b). The Serre-Tate lifting theorem (see the appendix to [Dr]) gives a bijection between (b) and (c). $\square$

Proposition 6.2 *Let p be a prime such that $p \nmid mN^+$, and let $t \in \mathcal{X}(\overline{\mathbb{F}}_p)$.*

(a) If $p \nmid N^-$ then $\hat{\mathcal{X}}_t \cong \operatorname{Spf} W[[u]]$.

(b) If $p \mid N^-$ and i is of type 1 or 2 then $\hat{\mathcal{X}}_t \cong \operatorname{Spf} W[[u]]$. If $p \mid N^-$ and i is of type 3 then $\hat{\mathcal{X}}_t \cong \operatorname{Spf} W[[u, v]]/(uv - p)$.

Proof: (a) In this case $\hat{A}_t$ has multiplication by $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \mathbb{M}_2(\mathbb{Z}_p)$ and must therefore be isomorphic to a product $G_0 \times G_0$, where G_0 is a formal group of dimension 1 and height 2 over $\overline{\mathbb{F}}_p$. By the same reasoning any deformation of $\hat{A}_t$ with multiplication by $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$ has the form $G \times G$, where G is a deformation of G_0, and conversely any deformation G of G_0 gives a unique deformation $G \times G$ of $\hat{A}_t$ with multiplication by $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$. Therefore deformations of the pair $(\hat{A}_t, \hat{\imath}_t)$ are equivalent to deformations of G_0. In [LT] it is proved that the universal deformation space of the formal group G_0 is $\operatorname{Spf} W_p[[u]]$. Hence by Lemma 6.1 we get $\hat{\mathcal{X}}_t \cong \operatorname{Spf} W_p[[u]]$.

(b) In this case $\hat{A}_t$ has multiplication by $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p \cong \hat{\mathcal{O}}_{1,p}$. We use Drinfeld's theory in [Dr] to interpret the formal scheme $\hat{\mathcal{H}}_p$ (the "p-adic upper half-plane") as a moduli space for rigidified formal groups of dimension 2 and height 4 with a special action by $\hat{\mathcal{O}}_{1,p}$. Associated to the pair $(\hat{A}_t, \hat{\imath}_t)$ is an equivalence class of closed points in $\hat{\mathcal{H}}_p$. The formal neighborhood of any of these points is a universal deformation space for $(\hat{A}_t, \hat{\imath}_t)$, and hence also for (A_t, i_t, Z_t, β_t).

To determine the structure of this formal neighborhood we use the explicit description of $\hat{\mathcal{H}}_p$ found in [Te, pp. 650–51] and [BC, I, §3]. The special fiber of $\hat{\mathcal{H}}_p$ is an infinite tree consisting of projective lines which meet transversely at their $\mathbb{F}_p$-rational points. The formal neighborhood of a closed point in $\hat{\mathcal{H}}_p$ takes two different forms, depending on whether or not it is a crossing point of the special fiber of $\hat{\mathcal{H}}_p$. To distinguish the crossing points from the other points in the special fiber of $\hat{\mathcal{H}}$ we let π be an element of $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$ such that $\pi^2 = p$. Then π is a generator for the maximal ideal of $\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p$, and π induces an endomorphism of $\operatorname{Lie}(\hat{A}_t)$ whose square is zero. If i_t is of type 1 or 2 then π induces a non-trivial endomorphism of $\operatorname{Lie}(\hat{A}_t)$. In this case t is not a crossing point of the special fiber of $\hat{\mathcal{H}}_p$, and [Te, p. 650] gives $\hat{\mathcal{X}}_t \cong \operatorname{Spf} W_p[[u]]$. If i_t is of type 3 then π induces the zero map on $\operatorname{Lie}(\hat{A}_t)$. In this case t is a crossing point of the special fiber of $\hat{\mathcal{H}}_p$, and by [Te, p. 650] we have $\hat{\mathcal{X}}_t \cong \operatorname{Spf} W_p[[u, v]]/(uv - p)$. $\square$

The following consequence of Proposition 6.2 is presumably well-known:

Corollary 6.3 $\mathcal{X} \otimes \mathbb{Z}[1/mN^+]$ *is a regular scheme.*

Let R be a complete noetherian local ring with residue field $\overline{\mathbb{F}}_p$ and let $(\hat{A}, \hat{\imath})$ be a deformation of $(\hat{A}_t, \hat{\imath}_t)$ defined over R. The reduction map $R \to \overline{\mathbb{F}}_p$ induces inclusions $\mathrm{End}(\hat{A}) \subset \mathrm{End}(\hat{A}_t)$ and $\mathrm{End}(\hat{A}, \hat{\imath}) \subset \mathrm{End}(\hat{A}_t, \hat{\imath}_t)$. Let $(\underline{\hat{A}}, \underline{\hat{\imath}})$ be a universal deformation of $(\hat{A}_t, \hat{\imath}_t)$ defined over $\hat{\mathcal{X}}_t$, and let S be a subset of $\mathrm{End}(\hat{A}_t, \hat{\imath}_t)$. We define $\hat{\mathcal{X}}_t(S)$ to be the largest formal subscheme of $\hat{\mathcal{X}}_t$ such that $S \subset \mathrm{End}(\underline{\hat{A}} \times_{\hat{\mathcal{X}}_t} \hat{\mathcal{X}}_t(S))$. The following facts are easily verified:

Lemma 6.4 *(a) $\hat{\mathcal{X}}_t(S)$ is closed in $\hat{\mathcal{X}}_t$.*
(b) $\hat{\mathcal{X}}_t(S_1 \cup S_2) = \hat{\mathcal{X}}_t(S_1) \cap \hat{\mathcal{X}}_t(S_2)$.
(c) Let $\mathbb{Z}_p[S]$ be the $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(\hat{A}_t, \hat{\imath}_t)$ generated by S. If $S \subset S' \subset \mathbb{Z}_p[S]$ then $\hat{\mathcal{X}}_t(S') = \hat{\mathcal{X}}_t(S)$.

Let $\gamma_1, \gamma_2 \in \mathrm{End}(\hat{A}_t, \hat{\imath}_t)$. The intersection multiplicity $\alpha_p(\gamma_1, \gamma_2)$ of $\hat{\mathcal{X}}_t(\gamma_1)$ with $\hat{\mathcal{X}}_t(\gamma_2)$ is defined to be the W_p-length of the coordinate ring of $\hat{\mathcal{X}}_t(\gamma_1) \cap \hat{\mathcal{X}}_t(\gamma_2) = \hat{\mathcal{X}}_t(\{\gamma_1, \gamma_2\})$. To compute the arithmetic intersection numbers of our divisors we need to evaluate $\alpha_p(\gamma_1, \gamma_2)$ for certain $\gamma_1, \gamma_2 \in \mathrm{End}(\hat{A}_t, \hat{\imath}_t)$.

Assume first that p is not ramified in Δ. Then $\hat{\mathcal{X}}_t$ is a universal deformation space for a formal group G_0 over $\overline{\mathbb{F}}_p$ of dimension 1 and height 2. By Proposition 4.3 we have $\mathrm{End}(\hat{A}_t, \hat{\imath}_t) \cong \mathrm{End}(G_0) \cong \hat{\mathcal{O}}_{1,p}$. The intersection multiplicity $\alpha_p(\gamma_1, \gamma_2)$ may be computed using the formulas in [GK, Prop. 5.4]. In order to state these formulas we define a quadratic form over $\mathbb{Z}_p$,

$$Q(x, y, z) = \mathrm{Nr}(x + y\gamma_1 + z\gamma_2), \tag{6.1}$$

where Nr is the reduced norm form on $\mathrm{End}(\hat{A}_t, \hat{\imath}_t) \cong \hat{\mathcal{O}}_{1,p}$. We wish to define invariants a_1, a_2, a_3 of Q. If $p > 2$ we diagonalize Q over $\mathbb{Z}_p$ and define $a_1 \le a_2 \le a_3$ to be the p-adic valuations of the coefficients of the diagonal form of Q. If $p = 2$ the definition of a_1, a_2, a_3 is more complicated and may be found in [GK, §4]. In either case we have $a_1 = 0$, so by [GK, Prop. 5.4] we get

$$\alpha_p(\gamma_1, \gamma_2) = \begin{cases} \dfrac{a_3 - a_2 + 1}{2} p^{a_2/2} + \displaystyle\sum_{i=0}^{(a_2-2)/2} (a_2 + a_3 - 4i)p^i & \text{if } a_2 \equiv 0 \pmod 2, \\ \displaystyle\sum_{i=0}^{(a_2-1)/2} (a_2 + a_3 - 4i)p^i & \text{if } a_2 \equiv 1 \pmod 2. \end{cases} \tag{6.2}$$

Note that the intersection multiplicity depends only on the $\mathbb{Z}_p$-isometry class of Q, and not on the particular γ_1, γ_2 that were used to define Q. Therefore we may write $\alpha_p(Q) = \alpha_p(\gamma_1, \gamma_2)$.

Now suppose that $p \mid N^-$ is ramified in Δ. We may assume without loss of generality that i_t is of type 1. By Proposition 4.3 we see that $\mathrm{End}(\hat{A}_t, \hat{\imath}_t) \cong \mathrm{End}(A_t, i_t, Z_t) \otimes \mathbb{Z}_p$ is a local Eichler order of type $(1, p^2)$. Suppose that for $j = 1, 2$ there are embeddings of $\mathcal{O}_{D_j}$ into $\mathrm{End}(A_t, i_t, Z_t)$ with image $\mathbb{Z}[\gamma_j]$. Using (3.3) we may assume that $p \nmid D_1$. It follows then from Remark 2.5 that $\mathcal{O}_{D_1} \otimes \mathbb{Z}_p \cong \mathbb{Z}_p[\gamma_1] \cong \mathbb{Z}_{p^2}$. Let $(\hat{A}, \hat{\imath})$ be a deformation

of $(\hat{A}_t, \hat{\imath}_t)$ such that $\gamma_1 \in \mathrm{End}(\hat{A}, \hat{\imath})$. Since i_t is of type 1, $\mathrm{End}(\hat{A})$ contains a subring which is conjugate in $\mathrm{End}(\hat{A}_t) \cong \mathbb{M}_2(\hat{\mathcal{O}}_{1,p})$ to

$$\left\{ \begin{bmatrix} a & b \\ pc & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_{p^2} \right\}. \tag{6.3}$$

Therefore we are in a situation much like the case where p is unramified in Δ: There is a formal group G_0 over $\overline{\mathbb{F}}_p$ of dimension 1 and height 2, a map $\tau : \mathbb{Z}_{p^2} \to \mathrm{End}(G_0)$, and a deformation G of G_0 such that $\hat{A} \cong G \times G$ and $\tau(\mathbb{Z}_{p^2}) \subset \mathrm{End}(G)$. Conversely, any deformation G of G_0 such that $\tau(\mathbb{Z}_{p^2}) \subset \mathrm{End}(G)$ gives a deformation $(\hat{A}, \hat{\imath})$ of $(\hat{A}_t, \hat{\imath}_t)$ such that $\gamma_1 \in \mathrm{End}(\hat{A}, \hat{\imath})$. It follows that $\hat{\mathcal{X}}_t(\gamma_1)$ is isomorphic to the universal deformation space of the formal $\mathbb{Z}_{p^2}$-module (G_0, τ). It is proved in [Gr] that the universal deformation of (G_0, τ) is the canonical lifting $\underline{G}$ of G_0 associated to τ, which is defined over W_p. Therefore we have $\hat{\mathcal{X}}_t(\gamma_1) \cong \mathrm{Spf}\, W_p$ and $\hat{\mathcal{X}}_t(\{\gamma_1, \gamma_2\}) \cong \mathrm{Spf}(W_p/(p^{k+1}))$ for some $k \geq 0$.

To determine k we use an indirect argument. Let $(\hat{A}, \hat{\imath})$ be the restriction of $(\underline{\hat{A}}, \underline{\hat{\imath}})$ to $\hat{\mathcal{X}}_t(\gamma_1) \cong \mathrm{Spf} W_p$, and for $n \geq 0$ let $(\hat{A}_n, \hat{\imath}_n)$ be the restriction of $(\underline{\hat{A}}, \underline{\hat{\imath}})$ to $\mathrm{Spf}(W_p/(p^{n+1}))$. Then $\hat{A}_n \cong G_n \times G_n$, where $G_n = \underline{G} \otimes (W_p/p^{n+1}W_p)$. By [Gr, Prop. 3.3] we have

$$\mathrm{End}(G_n) = \tau(\mathbb{Z}_{p^2}) + p^n \hat{\mathcal{O}}_{1,p}. \tag{6.4}$$

Using (4.5) we get

$$\mathrm{End}(\hat{A}_n, \hat{\imath}_n) = \left\{ \begin{bmatrix} a & b\pi \\ pb\pi & a \end{bmatrix} : a \in \tau(\mathbb{Z}_{p^2}),\ b \in p^n \tau(\mathbb{Z}_{p^2}) \right\}. \tag{6.5}$$

It follows that $\mathrm{End}(\hat{A}_n, \hat{\imath}_n)$ is a local Eichler order of type $(1, p^{2n+2})$.

Recall that k is the largest value of n such that $\mathbb{Z}_p[\gamma_1, \gamma_2]$ is contained in $\mathrm{End}(\hat{A}_n, \hat{\imath}_n)$. Since $\mathbb{Z}_p[\gamma_1, \gamma_2]$ and $\mathrm{End}(\hat{A}_n, \hat{\imath}_n)$ are both local Eichler orders which contain $\mathbb{Z}_p[\gamma_1] \cong \mathbb{Z}_{p^2}$, this implies $\mathbb{Z}_p[\gamma_1, \gamma_2] = \mathrm{End}(\hat{A}_k, \hat{\imath}_k)$. It follows that the reduced discriminant of $\mathbb{Z}_p[\gamma_1, \gamma_2]$ is p^{2k+2}. Let δ denote the reduced discriminant of the global order $\mathbb{Z}[\gamma_1, \gamma_2]$. Then we have $v_p(\delta) = 2k+2$, and hence

$$\alpha_p(\gamma_1, \gamma_2) = k + 1 = \frac{1}{2} \cdot v_p(\delta). \tag{6.6}$$

The following lemma will be used in determining the relationship between $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ and the divisor $\mathcal{P}_{D,\pm b}$.

Lemma 6.5 *Let G_0 be a formal group of dimension 1 and height 2 over $\overline{\mathbb{F}}_p$ and let $\hat{\mathcal{U}}$ be a universal deformation space for G_0. Let R_p be an order in a quadratic extension K of $\mathbb{Q}_p$, and let $\phi : R_p \to \mathrm{End}(G_0)$ be a ring homomorphism. Then $\hat{\mathcal{U}}(\phi(R_p))$ is reduced.*

Proof: We have $R_p = \mathbb{Z}_p + \mathbb{Z}_p a$ for some $a \in R_p$, so by Lemma 6.4(a) we get $\hat{\mathcal{U}}(\phi(a)) = \hat{\mathcal{U}}(\phi(R_p))$. The subscheme $\hat{\mathcal{U}}(\phi(a))$ of $\hat{\mathcal{U}} \cong \mathrm{Spf}\, W[[u]]$ is defined by an equation of the

form $f(u)-u=0$, where $f(u)\in W_p[[u]]$ (cf. [LT, p. 58]). Let p^k be the conductor of the order R_p. Then the Weierstrass degree of $f(u)-u$ is computed in [Ke1, Th. 1.1] to be

$$\begin{aligned} p^k+2p^{k-1}+\cdots+2p+2 \quad &\text{if } K/\mathbb{Q}_p \text{ is unramified,}\\ 2p^k+2p^{k-1}+\cdots+2p+2 \quad &\text{if } K/\mathbb{Q}_p \text{ is ramified.} \end{aligned} \tag{6.7}$$

The power series $f(u)-u$ is divisible by an irreducible factor corresponding to a quasi-canonical lifting of level l for each $0\le l\le k$. The Weierstrass degrees of these factors are computed in [Gr, Prop. 5.3] to be

$$\begin{array}{cl} 1 & \text{if } K/\mathbb{Q}_p \text{ is unramified and } l=0,\\ p^l+p^{l-1} & \text{if } K/\mathbb{Q}_p \text{ is unramified and } l\ge 1,\\ 2p^l & \text{if } K/\mathbb{Q}_p \text{ is ramified.} \end{array} \tag{6.8}$$

Comparing Weierstrass degrees we find that $f(u)-u$ is the product of the quasi-canonical lifting factors for $0\le l\le k$ and a unit power series. Since the quasi-canonical lifting factors are irreducible and have different degrees, the quotient $W_p[[u]]/(f(u)-u)$ is reduced, as claimed. $\square$

Choose D and $b=(b_l)_{l|N}$ as in §2. Let $t\in\mathcal{X}(\overline{\mathbb{F}}_p)$ be such that $(A_t,i_t,Z_t)\in\mathcal{T}_p$. We wish to consider the restriction $\hat{\mathcal{P}}_{D,\pm b}$ of the divisor $\mathcal{P}_{D,\pm b}$ to the completion $\hat{\mathcal{X}}_t$ of $\mathcal{X}\otimes W_p$ at t. Let M_p denote the field of fractions of W_p and replace X with $X\otimes M_p$ and $P_{D,\pm b}$ with $P_{D,\pm b}\otimes M_p$. Then $\mathcal{P}_{D,\pm b}\otimes W_p$ is the closure of $P_{D,\pm b}\otimes M_p$ in $\mathcal{X}\otimes W_p$. For each x in the support of $P_{D,\pm b}\otimes M_p$ define $\hat{x}$ to be the closure of x in $\hat{\mathcal{X}}_t$. Then we have $\hat{\mathcal{P}}_{D,\pm b}=\sum(\hat{x})$, where the sum is taken over all points x in the support of $P_{D,\pm b}\otimes M_p$ such that $t\in\overline{x}$.

Let $(\underline{A},\underline{i},\underline{Z})$ be a universal deformation of (A_t,i_t,Z_t) defined over $\hat{\mathcal{X}}_t$, and let $\{\psi_l\}_{l|pN}$ be the orientation on $\mathrm{End}(A_t,i_t,Z_t)$ induced by $\{\phi_l\}_{l|N}$. Define a b-embedding to be a ring homomorphism $\sigma:\mathcal{O}_D\to\mathrm{End}(A_t,i_t,Z_t)$ such that $\psi_l\circ\sigma(\sqrt{D})=b_l$ for all $l\mid N$. Say that σ is a $\pm b$-embedding if σ is either a b-embedding or a $(-b)$-embedding. The relation between $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ and $\hat{\mathcal{P}}_{D,\pm b}$ is given by the following lemma.

Lemma 6.6 *(a) If σ is a $\pm b$-embedding then $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ is contained in the support of $\hat{\mathcal{P}}_{D,\pm b}$.*

(b) Every irreducible component of the support of $\hat{\mathcal{P}}_{D,\pm b}$ lies in $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ for some $\pm b$-embedding σ.

(c) Let σ,σ' be $\pm b$-embeddings. Then the components in the support of $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ and $\hat{\mathcal{X}}_t(\sigma'(\mathcal{O}_D))$ are all different unless $\sigma(\mathcal{O}_D)=\sigma'(\mathcal{O}_D)$.

Proof: (a) Suppose $p\nmid N^-$. In the proof of Proposition 6.2 (a) we showed that $\hat{\mathcal{X}}_t$ is a universal deformation space for a formal group G_0 of dimension 1 and height 2. The map σ induces an embedding of $\mathcal{O}_D\otimes\mathbb{Z}_p$ into $\mathrm{End}(G_0)$, and $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ is defined by the requirement that the image of this embedding should lift. It follows from Lemma 6.5

that $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ is reduced. Therefore it suffices to show that the generic fibers of the irreducible components of $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ all have characteristic 0. But this follows from the fact that the reduction (mod p) of a universal deformation of G_0 has endomorphism ring $\mathbb{Z}_p$ (cf. [Ke1, Th. 1.1]).

Suppose $p \mid N^-$. It follows from assumption (3.2) and Remark 2.5 that $\mathcal{O}_D \otimes \mathbb{Z}_p$ is the ring of integers in a quadratic extension of $\mathbb{Q}_p$. Let $(\hat{A}, \hat{\imath})$ be a deformation of $(\hat{A}_t, \hat{\imath}_t)$ such that $\sigma(\mathcal{O}_D) \otimes \mathbb{Z}_p \subset \mathrm{End}(\hat{A}, \hat{\imath})$. Since $\mathrm{End}(\hat{A}, \hat{\imath})$ is contained in $\mathrm{End}(\hat{A}_t, \hat{\imath}_t)$, which is a local Eichler order of type $(1, p^2)$, we must have $\mathcal{O}_D \otimes \mathbb{Z}_p \cong \mathbb{Z}_{p^2}$. Therefore $\hat{A}$ has multiplication by

$$(\mathcal{O}_{1,N^-} \otimes \mathbb{Z}_p) \otimes (\mathcal{O}_D \otimes \mathbb{Z}_p) \cong \hat{\mathcal{O}}_{1,p} \otimes \mathbb{Z}_{p^2} \tag{6.9}$$

$$\cong \left\{ \begin{bmatrix} a & b \\ pc & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_{p^2} \right\}. \tag{6.10}$$

It follows that $\hat{A} \cong G \times G$, where G is a deformation of G_0. Since $\sigma(\mathcal{O}_D) \otimes \mathbb{Z}_p$ commutes with the image of $\hat{\imath}$, the map σ induces an embedding $\tau : \mathcal{O}_D \otimes \mathbb{Z}_p \to \mathrm{End}(G_0)$ such that $\tau(\mathcal{O}_D \otimes \mathbb{Z}_p) \subset \mathrm{End}(G)$. Conversely, any deformation G of G_0 such that $\tau(\mathcal{O}_D \otimes \mathbb{Z}_p) \subset \mathrm{End}(G)$ gives a deformation $(\hat{A}, \hat{\imath})$ of $(\hat{A}_t, \hat{\imath}_t)$ such that $\sigma(\mathcal{O}_D) \otimes \mathbb{Z}_p \subset \mathrm{End}(\hat{A}, \hat{\imath})$. The maximal deformation G of G_0 such that $\tau(\mathcal{O}_D \otimes \mathbb{Z}_p) \subset \mathrm{End}(G)$ is the canonical lifting of G_0 associated to τ. By [Gr, Prop. 2.1] the canonical lifting is defined over an integral domain of characteristic 0. As above this implies that $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ is contained in the support of $\hat{\mathcal{P}}_{D,\pm b}$.

(b) Let $\hat{y}$ be an irreducible component of $\hat{\mathcal{P}}_{D,\pm b}$, and let (A_y, i_y, Z_y) be the triple corresponding to y. Then there are two embeddings $\rho_y, \overline{\rho}_y : \mathcal{O}_D \to \mathrm{End}(A_y, i_y, Z_y)$, where $\overline{\rho}_y$ is ρ_y composed with complex conjugation on $\mathcal{O}_D$. Let

$$j : \mathrm{End}(A_y, i_y, Z_y) \longrightarrow \mathrm{End}(A_t, i_t, Z_t) \tag{6.11}$$

be the natural embedding and define $\sigma = j \circ \rho_y$. Then $\hat{y}$ is contained in $\mathcal{X}(\sigma(\mathcal{O}_D))$. We need to show that $\sigma : \mathcal{O}_D \to \mathrm{End}(A_t, i_t, Z_t)$ is a $\pm b$-embedding.

Suppose $l \mid N$ with $l \neq p$. Let $T_l(A_y)$, $T_l(A_t)$ be the l-adic Tate modules of A_y, A_t, and let $U_l^y \supset T_l(A_y)$, $U_l^t \supset T_l(A_t)$ be the lattices which correspond to the l-primary parts of Z_y, Z_t. Then there is a natural $(\mathcal{O}_{N^+,N^-} \otimes \mathbb{Z}_l)$-linear isomorphism $\nu : U_l^y \to U_l^t$. For $\gamma \in \mathrm{End}(A_y, i_y, Z_y)$ let $\tilde{\gamma}$ denote the endomorphism of U_l^y induced by γ, and let $\widetilde{j(\gamma)}$ denote the endomorphism of U_l^t induced by $j(\gamma)$. Then we have $\nu \circ \tilde{\gamma} = \widetilde{j(\gamma)} \circ \nu$. It now follows from the definitions of ψ_l and ω_l that $\psi_l \circ j(\gamma) = \omega_l(\gamma)$. Hence

$$\psi_l \circ \sigma = \psi_l \circ j \circ \rho_y = \omega_l \circ \rho_y. \tag{6.12}$$

Suppose $p \mid N^-$. It follows from Remark 2.5 that p does not split in K, and it follows from the fact that i_t has type 1 or 2 that p is not ramified in K. Thus p is inert in K. Hence by Proposition 2.1, $T_p(A_y)/pT_p(A_y)$ is a vector space of dimension 2 over $\mathcal{R}/p\mathcal{R} \cong \mathbb{F}_{p^2}$. Furthermore, the representation of $\mathrm{End}(A_y)$ on $T_p(A_y)/pT_p(A_y)$ is

equivalent to the representation of $\mathrm{End}(A_y)$ on $\mathrm{Lie}(A_t)$ induced by j. Therefore by the constructions of ψ_p and ω_p we get $\psi_p \circ j \equiv \omega_p \pmod{p}$. It follows that

$$\psi_p \circ \sigma \equiv \psi_p \circ j \circ \rho_y \equiv \omega_p \circ \rho_y \pmod{p}. \tag{6.13}$$

Since y lies in the support of $P_{D,\pm b}$, it follows from (6.12) and (6.13) that σ is a $\pm b$-embedding.

(c) Suppose $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ and $\hat{\mathcal{X}}_t(\sigma'(\mathcal{O}_D))$ have a component in common. Then by (a) this component is contained in the support of $\hat{\mathcal{P}}_{D,\pm b}$. Let y be the generic point of this component. Then y has characteristic 0, so by Proposition 2.1, $\mathrm{End}(A_y, i_y, Z_y)$ is an order in $\mathbb{Q}(\sqrt{D})$. Since $\sigma(\mathcal{O}_D)$ and $\sigma'(\mathcal{O}_D)$ both lie in $\mathrm{End}(A_y, i_y, Z_y)$, we must have $\sigma(\mathcal{O}_D) = \sigma'(\mathcal{O}_D)$. □

Remark 6.7 Suppose p is ramified in Δ and p divides the conductor c of $\mathcal{O}_D$. Then $\hat{\mathcal{X}}_t(\sigma(\mathcal{O}_D))$ contains the subscheme of $\hat{\mathcal{X}}_t$ defined by the ideal (p). Therefore (a) and (c) of Lemma 6.6 are false if $p \mid c$. (However, (b) holds even if $p \mid c$.)

Proposition 6.8 *We have $\hat{\mathcal{P}}_{D,\pm b} = \frac{1}{2} \cdot \sum \mathcal{X}_t(\sigma(\mathcal{O}_D))$, where the sum is taken over all $\pm b$-embeddings $\sigma : \mathcal{O}_D \to \mathrm{End}(A_t, i_t, Z_t)$.*

Proof: This follows from Lemma 6.6. The factor $\frac{1}{2}$ arises because σ has the same image as $\overline{\sigma}$, where $\overline{\sigma}$ is σ composed with complex conjugation. □

7 Completion of the proofs

In this section we use the results proved in §4–§6 to compute $\langle \mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2} \rangle_{\mathcal{X}}$ and $\langle \mathcal{P}_{D_1} \cdot \mathcal{P}_{D_2} \rangle_{\mathcal{X}}$. We start by deriving a preliminary formula for $\langle \mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2} \rangle_{\mathcal{X}}$.

It follows from (4.1) that there are integers $(\mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2})_p$ such that

$$\langle \mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2} \rangle_{\mathcal{X}} = \sum_{p<\infty} (\mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2})_p \cdot \log p. \tag{7.1}$$

The quantity $(\mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2})_p$ can be interpreted as an intersection multiplicity on $\mathcal{X} \otimes W_p$. Let $\mathcal{I}_p$ denote the set of points $t \in \mathcal{X}(\overline{\mathbb{F}}_p)$ such that $(A_t, i_t, Z_t) \in \mathcal{T}_p$. Then the support of the intersection of $\mathcal{P}_{D_1,\pm b_1} \otimes W_p$ with $\mathcal{P}_{D_2,\pm b_2} \otimes W_p$ on $\mathcal{X} \otimes W_p$ is contained in $\mathcal{I}_p$. Therefore we have

$$(\mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2})_p = \sum_{t \in \mathcal{I}_p} (\mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2})_t. \tag{7.2}$$

Given embeddings σ_1, σ_2 of $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}$ into $\mathrm{End}(A_t, i_t, Z_t)$, set $\epsilon_j = \sigma_j(\sqrt{D_j})$ and $\gamma_j = (D_j + \epsilon_j)/2$ for $j = 1, 2$. Then $\gamma_j \in \mathrm{End}(A_t, i_t, Z_t)$ and $\sigma_j(\mathcal{O}_{D_j}) = \mathbb{Z}_p[\gamma_j]$. Hence by Lemma 6.4(c) and Proposition 6.8 we have

$$(\mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2})_t = \frac{1}{4} \cdot \sum_{\sigma_1, \sigma_2} \alpha_p(\gamma_1, \gamma_2), \tag{7.3}$$

where the sum is taken over all pairs (σ_1, σ_2) such that σ_j is a $\pm b_j$-embedding.

It follows from Proposition 4.3 that $\mathrm{End}(A_t, i_t, Z_t)$ is isomorphic to an Eichler order of type (N^+, pN^-) in the quaternion algebra $\Delta(p)$ over $\mathbb{Q}$. Therefore $\mathrm{End}(A_t, i_t, Z_t)$ has reduced discriminant $N^+ \cdot pN^- = pN$. Let $n = \frac{1}{2} \cdot \mathrm{Tr}(\epsilon_1\epsilon_2) = 2\mathrm{Tr}(\gamma_1\gamma_2) - D_1D_2$, where Tr is the reduced trace from $\Delta(p)$ to $\mathbb{Q}$. Since σ_j is a $\pm b_j$-embedding we have $n \equiv \pm h \pmod{2N}$, where h is determined by (3.8) and (3.9). Since $n \equiv D_1D_2 \pmod 2$, the ring S_n from §1 is defined. In fact $S_n \cong \mathbb{Z}[\gamma_1, \gamma_2]$ is isomorphic to a suborder of $\mathrm{End}(A_t, i_t, Z_t)$, and hence $B_n \cong \Delta(p)$ and pN divides the reduced discriminant $\delta_n = (n^2 - D_1D_2)/4$ of S_n. Since $\Delta(p)$ is ramified at ∞ we have $\delta_n < 0$, and hence $n^2 < D_1D_2$. It follows that $p \leq |\delta_n|/N \leq D_1D_2/4N$, so by assumption C3 in (2.2) we have $p \nmid m$.

The ternary quadratic form $\mathrm{Nr}(x + y\gamma_1 + z\gamma_2)$ over $\mathbb{Z}$ is equal to the form $Q_n(x, y, z)$ defined in (1.5). In §6 we saw that the intersection number $\alpha_p(\gamma_1, \gamma_2)$ depends only on the $\mathbb{Z}_p$-isometry class of Q_n and not on the choice of γ_1, γ_2. Therefore we may write $\alpha_p(Q_n) = \alpha_p(\gamma_1, \gamma_2)$. Using (7.2) and (7.3) we get

$$(\mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2})_p = \sum_{t \in \mathcal{I}_p} \left(\frac{1}{4} \cdot \sum_{\sigma_1,\sigma_2} \alpha_p(\gamma_1, \gamma_2) \right) \tag{7.4}$$

$$= \frac{1}{4} \cdot \sum_{\substack{n^2 < D_1D_2 \\ n \equiv \pm h \ (2N)}} \left(\sum_{t \in \mathcal{I}_p} r_t(n, \pm b_1, \pm b_2) \right) \cdot \alpha_p(Q_n), \tag{7.5}$$

where $r_t(n, \pm b_1, \pm b_2)$ denotes the number of pairs (σ_1, σ_2) of $\pm b_j$-embeddings of $\mathcal{O}_{D_1}, \mathcal{O}_{D_2}$ into $\mathrm{End}(A_t, i_t, Z_t)$ such that $\frac{1}{2} \cdot \mathrm{Tr}(\epsilon_1\epsilon_2) = n$.

We wish to interpret the inner sum of (7.5) as counting embeddings of the ring S_n into Eichler orders. Let $\mathcal{E} \subset \Delta(p)$ be an Eichler order of type (N^+, pN^-), and let

$$N(\mathcal{E}) = \{\beta \in \Delta(p)^\times : \beta\mathcal{E}\beta^{-1} = \mathcal{E}\}. \tag{7.6}$$

Then $N(\mathcal{E})/\mathbb{Q}^\times\mathcal{E}^\times$ acts freely on the set of orientations on $\mathcal{E}$; the orbits of this action are the isomorphism classes of orientations for $\mathcal{E}$. Let $v_\mathcal{E} = |N(\mathcal{E})/\mathbb{Q}^\times\mathcal{E}^\times|$ and recall that r is the number of distinct primes dividing N. Thus if $p \nmid N^-$ then $\mathcal{E}$ has 2^{r+1} orientations, and hence $2^{r+1}/v_\mathcal{E}$ isomorphism classes of orientations, while if $p \mid N^-$ then $\mathcal{E}$ has 2^r orientations, and hence $2^r/v_\mathcal{E}$ isomorphism classes of orientations. By Proposition 4.3, $\mathrm{End}(A_t, i_t, Z_t)$ is an Eichler order of type (N^+, pN^-) for each $t \in \mathcal{I}_p$. Let $\mathcal{I}_\mathcal{E}$ denote the set of $t \in \mathcal{I}_p$ such that $\mathrm{End}(A_t, i_t, Z_t) \cong \mathcal{E}$. Let $\mathcal{C}_p$ be a set of representatives for the isomorphism classes of Eichler orders of type (N^+, pN^-). Then

$$\sum_{t \in \mathcal{I}_p} r_t(n, \pm b_1, \pm b_2) = \sum_{\mathcal{E} \in \mathcal{C}_p} \sum_{t \in \mathcal{I}_\mathcal{E}} r_t(n, \pm b_1, \pm b_2). \tag{7.7}$$

Given an orientation $\{\mu_l\}_{l|pN}$ on the Eichler order $\mathcal{E}$, say that the homomorphism $\tau : S_n \to \mathcal{E}$ is a (b_1, b_2)-embedding if $\mu_l \circ \tau(e_j) \equiv (b_j)_l \pmod{l^{a_l}}$ for $j = 1, 2$ and all $l \mid N$,

where $a_l = v_l(2N)$. Say that τ is a $(\pm b_1, \pm b_2)$-embedding if τ is a (b_1, b_2)-, $(-b_1, b_2)$-, $(b_1, -b_2)$-, or $(-b_1, -b_2)$-embedding. Let

$$\mathcal{T}_{\mathcal{E}} = \{(A, i, Z) \in \mathcal{T}_p : \operatorname{End}(A, i, Z) \cong \mathcal{E}\} \tag{7.8}$$

and let $\tilde{\mathcal{I}}_{\mathcal{E}}$ be a subset of $\mathcal{I}_{\mathcal{E}}$ such that each $(A, i, Z) \in \mathcal{T}_{\mathcal{E}}$ is isomorphic to (A_t, i_t, Z_t) for exactly one $t \in \tilde{\mathcal{I}}_{\mathcal{E}}$.

Let $r_{\mathcal{E}}(n)$ denote the number of embeddings of S_n into $\mathcal{E}$. Suppose $p \nmid N^-$. Then each homomorphism $\tau : S_n \to \mathcal{E}$ is a $(\pm b_1, \pm b_2)$-embedding for four different orientations on $\mathcal{E}$. Hence the set

$$\Upsilon_n = \{(\tau : S_n \to \mathcal{E}, \{\mu_l\}_{l|pN}) : \tau \text{ is a } (\pm b_1, \pm b_2)\text{-embedding w. r. t. } \{\mu_l\}_{l|pN}\} \tag{7.9}$$

has cardinality $4r_{\mathcal{E}}(n)$. It follows from Proposition 5.1(a) that the isomorphism classes of orientations on $\mathcal{E}$ correspond to elements $t \in \tilde{\mathcal{I}}_{\mathcal{E}}$. This allows us to count the elements of Υ_n in a different manner, and gives the formula

$$4r_{\mathcal{E}}(n) = v_{\mathcal{E}} \cdot \sum_{t \in \tilde{\mathcal{I}}_{\mathcal{E}}} r_t(n, \pm b_1, \pm b_2). \tag{7.10}$$

Suppose $p \mid N$. Then by Proposition 5.1(b) each isomorphism class of orientations on $\mathcal{E}$ is represented by two different $t \in \tilde{\mathcal{I}}_{\mathcal{E}}$, and each homomorphism $\tau : S_n \to \mathcal{E}$ is a $(\pm b_1, \pm b_2)$-embedding for two orientations on $\mathcal{E}$. Hence (7.10) is valid in this case as well.

Let $(A, i, Z) \in \mathcal{T}_{\mathcal{E}}$. Then the group $\operatorname{Aut}(A, i, Z) \cong \mathcal{E}^\times$ acts freely on the set of level-m structures β on (A, i). By Remark 3.4 the pair (A, i) admits $2\eta(m)$ different level-m structures. Hence there are $2\eta(m)/2u_{\mathcal{E}}$ isomorphism classes of 4-tuples (A, i, Z, β), where $u_{\mathcal{E}} = \frac{1}{2} \cdot \#\mathcal{E}^\times$. It follows that

$$\sum_{t \in \mathcal{I}_{\mathcal{E}}} r_t(n, \pm b_1, \pm b_2) = \frac{\eta(m)}{u_{\mathcal{E}}} \cdot \sum_{t \in \tilde{\mathcal{I}}_{\mathcal{E}}} r_t(n, \pm b_1, \pm b_2). \tag{7.11}$$

Combining this formula with (7.10) we get

$$\sum_{t \in \mathcal{I}_{\mathcal{E}}} r_t(n, \pm b_1, \pm b_2) = \frac{4\eta(m) r_{\mathcal{E}}(n)}{u_{\mathcal{E}} v_{\mathcal{E}}}. \tag{7.12}$$

It follows from (7.7) and (7.12) that

$$\sum_{t \in \mathcal{I}_p} r_t(n, \pm b_1, \pm b_2) = 4\eta(m) \cdot \sum_{\mathcal{E} \in \mathcal{C}_p} \frac{r_{\mathcal{E}}(n)}{u_{\mathcal{E}} v_{\mathcal{E}}}. \tag{7.13}$$

By combining this formula with (7.5) and (7.1) we get the following formula for the arithmetic intersection number:

$$\langle \mathcal{P}_{D_1, \pm b_1} \cdot \mathcal{P}_{D_2, \pm b_2} \rangle_{\mathcal{X}} = \eta(m) \cdot \sum_{p < \infty} \left(\sum_{\substack{n^2 < D_1 D_2 \\ n \equiv \pm h \ (2N)}} \left(\sum_{\mathcal{E} \in \mathcal{C}_p} \frac{r_{\mathcal{E}}(n)}{u_{\mathcal{E}} v_{\mathcal{E}}} \right) \cdot \alpha_p(Q_n) \right) \cdot \log p. \tag{7.14}$$

To prove Theorem 3.2 we will evaluate the inner sum of (7.14) in terms of representation numbers of quadratic forms. To prove Theorem 3.6 we will evaluate the inner sum of (7.14) by counting Eichler orders.

Proof of Theorem 3.2: Recall that r denotes the number of distinct prime divisors of N. For each n such that $n^2 \equiv D_1D_2 \pmod{2N}$ there are 2^{r-1} pairs $(\pm b_1, \pm b_2)$ such that $n \equiv \pm h(\pm b_1, \pm b_2) \pmod{2N}$. If $r_{\mathcal{E}}(n) \neq 0$ then pN divides the reduced discriminant $\delta_n = (n^2 - D_1D_2)/4$ of S_n, so we have $n^2 \equiv D_1D_2 \pmod{4pN}$. Therefore by summing (7.14) over all 2^{2r-2} pairs $(\pm b_1, \pm b_2)$ we get

$$\langle \mathcal{P}_{D_1} \cdot \mathcal{P}_{D_2} \rangle_{\mathcal{X}} = 2^{r-1} \cdot \eta(m) \cdot \sum_{p<\infty} \left(\sum_{\substack{n^2 < D_1D_2 \\ n^2 \equiv D_1D_2 \ (4pN)}} \left(\sum_{\mathcal{E} \in \mathcal{C}_p} \frac{r_{\mathcal{E}}(n)}{u_{\mathcal{E}} v_{\mathcal{E}}} \right) \cdot \alpha_p(Q_n) \right) \cdot \log p. \quad (7.15)$$

Let $\mathcal{E} \in \mathcal{C}_p$ and let n be an integer such that $n^2 < D_1D_2$ and $n^2 \equiv D_1D_2 \pmod{4pN}$. Recall that $S_n = \mathbb{Z}[g_1, g_2]$ and define $L_n = \mathbb{Z} + \mathbb{Z}g_1 + \mathbb{Z}g_2 \subset S_n$. Then L_n is a quadratic space with the reduced norm form Nr. Let (τ, λ) be a pair consisting of a ring homomorphism $\tau : S_n \to \mathcal{E}$ and a unit $\lambda \in \mathcal{E}^\times$. Associated to this pair is an isometry $\lambda \cdot \tau|_{L_n}$ of L_n into $\mathcal{E}$. We claim that every isometry from L_n into $\mathcal{E}$ arises this way. Let $\nu : L_n \to \mathcal{E}$ be an isometry, and set $\lambda = \nu(1)$. Then $1 = \mathrm{Nr}(1) = \mathrm{Nr}(\lambda)$, so $\lambda \in \mathcal{E}^\times$. Define a $\mathbb{Z}$-linear map Let $\tau : S_n \to \mathcal{E}$ be the unique $\mathbb{Z}$-linear map such that $\tau(1) = 1$, $\tau(g_j) = \lambda^{-1}\nu(g_j)$, and $\tau(g_1g_2) = \tau(g_1)\tau(g_2)$. Set $\epsilon_j = \tau(e_j)$ for $j = 1, 2$. Since ν is an isometry we get $\epsilon_j^2 = -\mathrm{Nr}(\epsilon_j) = -\mathrm{Nr}(e_j) = D_j$ and

$$\begin{aligned} (\epsilon_1 + \epsilon_2)^2 &= -\mathrm{Nr}(\epsilon_1 + \epsilon_2) && (7.16) \\ &= -\mathrm{Nr}(e_1 + e_2) && (7.17) \\ &= (e_1 + e_2)^2, && (7.18) \end{aligned}$$

which implies $\epsilon_1\epsilon_2 + \epsilon_2\epsilon_1 = 2n$. It follows that

$$\tau \otimes \mathbb{Q} : S_n \otimes \mathbb{Q} \longrightarrow \mathcal{E} \otimes \mathbb{Q} \quad (7.19)$$

is an isomorphism of rings, and hence that τ is a ring homomorphism.

Since $\mathbb{Z}^3$ with the quadratic form Q_n is isometric to L_n, it follows from the preceding paragraph that $\mathcal{R}_{\mathcal{E}}(Q_n) = \#\mathcal{E}^\times \cdot r_{\mathcal{E}}(n) = 2u_{\mathcal{E}} r_{\mathcal{E}}(n)$. By Lemma 5.3 the number of proper self-isometries of $\mathcal{E}$ is $w_{\mathcal{E}} = 2u_{\mathcal{E}}^2 v_{\mathcal{E}}$. Therefore $r_{\mathcal{E}}(n)/u_{\mathcal{E}}v_{\mathcal{E}} = \mathcal{R}_{\mathcal{E}}(Q_n)/w_{\mathcal{E}}$. Hence by Proposition 5.2 we have

$$\sum_{\mathcal{E} \in \mathcal{C}_p} \frac{r_{\mathcal{E}}(n)}{u_{\mathcal{E}} v_{\mathcal{E}}} = \sum_{L \in \mathcal{G}_p} \frac{\mathcal{R}_L(Q_n)}{w_L}. \quad (7.20)$$

Substituting this formula into (7.15) gives Theorem 3.2. □

Proof of Theorem 3.6: Choose n such that $n^2 < D_1D_2$ and $n \equiv \pm h \pmod{2N}$. Since $\gcd(D_1, D_2) = 1$, it follows from §1 that S_n is an Eichler order of type (δ_n^+, δ_n^-), and it

follows from Remark 3.7 that $N^+ \mid \delta_n^+$ and $N^- \mid \delta_n^-$. Set $M_n^+ = \delta_n^+/N^+$, $M_n^- = \delta_n^-/N^-$, $m_p^+ = v_p(M_n^+)$, and $m_p^- = v_p(M_n^-)$, and define $L_{(p)}(s) = L_{p^{m_p^+},\, p^{m_p^-}}(s)$. Then we have

$$L_{M_n^+,M_n^-}(s) = \prod_{p<\infty} L_{(p)}(s) \tag{7.21}$$

$$L'_{M_n^+,M_n^-}(0) = \sum_{p<\infty} \left(\prod_{l\neq p} L_{(l)}(0) \right) L'_{(p)}(0). \tag{7.22}$$

Let p be a prime, and assume for now that $S_n \otimes \mathbb{Q} \cong \Delta(p)$. If $p \mid N^+$ then $p \nmid \delta_n^-$ by Remark 3.7, so this assumption implies $p \nmid N^+$. Let $\mathcal{E}_0$ be an Eichler order of type (N^+, pN^-). We may compute $r_{\mathcal{E}_0}(n)$ as the number of embeddings of $\mathcal{E}_0$ into $S_n \otimes \mathbb{Q}$ whose image contains S_n. Hence $r_{\mathcal{E}_0}(n)$ is the sum over all $\mathcal{E}$ such that $S_n \subset \mathcal{E} \subset S_n \otimes \mathbb{Q}$ of the number of isomorphisms of $\mathcal{E}_0$ onto $\mathcal{E}$. If $\mathcal{E}_0 \cong \mathcal{E}$ then there are $|N(\mathcal{E}_0)/\mathbb{Q}^\times| = u_{\mathcal{E}_0} v_{\mathcal{E}_0}$ such isomorphisms. We conclude that the number of Eichler orders $\mathcal{E}$ such that $S_n \subset \mathcal{E} \subset S_n \otimes \mathbb{Q}$ and $\mathcal{E} \cong \mathcal{E}_0$ is $r_{\mathcal{E}_0}(n)/u_{\mathcal{E}_0} v_{\mathcal{E}_0}$. It follows that the inner sum $\sum_{\mathcal{E}\in\mathcal{C}_p} r_{\mathcal{E}}(n)/u_{\mathcal{E}} v_{\mathcal{E}}$ of (7.14) is equal to the number of Eichler orders $\mathcal{E}$ of type (N^+, pN^-) such that $S_n \subset \mathcal{E} \subset S_n \otimes \mathbb{Q}$.

For each prime l set $e_l^+ = v_l(N^+)$, $e_l^- = v_l(pN^-)$, $d_l^+ = v_l(\delta_n^+)$, and $d_l^- = v_l(\delta_n^-)$. The number of Eichler orders in $S_n \otimes \mathbb{Q}$ of type (N^+, pN^-) which contain the Eichler order S_n can be computed as the product over l of the number c_l of local Eichler orders $\mathcal{E}_l$ of type $(l^{e_l^+}, l^{e_l^-})$ in $S_n \otimes \mathbb{Q}_l$ such that $\mathcal{E}_l \supset S_n \otimes \mathbb{Z}_l$. If $S_n \otimes \mathbb{Z}_l$ has type $(l^{d_l^+}, 1)$ then $c_l = d_l^+ - e_l^+ + 1 = L_{(l)}(0)$. If $S_n \otimes \mathbb{Z}_l$ has type $(1, l^{d_l^-})$ then $c_l = 1$ for $d_l^- \equiv e_l^- \pmod 2$ and $c_l = 0$ for $d_l^- \not\equiv e_l^- \pmod 2$, so once again we have $c_l = L_{(l)}(0)$. Since we are assuming $S_n \otimes \mathbb{Q} \cong \Delta(p)$ we get $c_p = 1$. Hence the total number of Eichler orders $\mathcal{E}$ of type (N^+, pN^-) in $\Delta(p)$ which contain S_n is

$$\sum_{\mathcal{E}\in\mathcal{C}_p} \frac{r_{\mathcal{E}}(n)}{u_{\mathcal{E}} v_{\mathcal{E}}} = \prod_{l\neq p} L_{(l)}(0). \tag{7.23}$$

Now suppose that $S_n \otimes \mathbb{Q}$ is not isomorphic to $\Delta(p)$. In this case the sum on the left side of (7.23) is 0, and there is at least one prime $l \neq p$ such that $v_l(M_n^-)$ is odd. It follows that $L_{(l)}(0) = 0$, so the right side of (7.23) is also 0. Hence (7.23) is valid for all primes p.

Suppose $p \nmid N$. Then the assumption $\gcd(D_1, D_2) = 1$ implies that, in the notation of (6.2), we have $a_2 = 0$ and $a_3 = v_p(\delta_n) = v_p(\delta_n^-)$. Therefore $\alpha_p(Q_n) = (v_p(\delta_n^-) + 1)/2$. On the other hand, if $p \mid N^-$ then by (6.6) we have $\alpha_p(Q_n) = v_p(\delta_n^-)/2$. In both cases we get $\alpha_p(Q_n) = (v_p(M_n^-) + 1)/2$. If $S_n \otimes \mathbb{Q} \cong \Delta(p)$ then $v_p(M_n^-)$ is odd. Therefore by (3.6) we get

$$\begin{aligned} L'_{(p)}(0) &= \frac{v_p(M_n^-) + 1}{2} \cdot \log p && (7.24)\\ &= \alpha_p(Q_n) \cdot \log p. && (7.25) \end{aligned}$$

It follows from (7.22), (7.23), and (7.25) that

$$L'_{M_n^+,M_n^-}(0) = \sum_{p<\infty} \left(\sum_{\mathcal{E}\in\mathcal{C}_p} \frac{r_{\mathcal{E}}(n)}{u_{\mathcal{E}} v_{\mathcal{E}}} \right) \cdot \alpha_p(Q_n) \cdot \log p. \tag{7.26}$$

Combining this formula with (7.14) we get

$$\langle \mathcal{P}_{D_1,\pm b_1} \cdot \mathcal{P}_{D_2,\pm b_2} \rangle_{\mathcal{X}} = \eta(m) \cdot \sum_{\substack{n^2<D_1D_2 \\ n\equiv \pm h\ (2N)}} L'_{M_n^+,M_n^-}(0), \tag{7.27}$$

which is Theorem 3.6. $\square$

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